ARTICLES

Quantum plasmas with or without a uniform magnetic field. I. General formalism and algebraic tails of correlations

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Large-distance quantum static correlations are investigated in a fluid of point charges interacting via Coulomb forces in the presence of a uniform magnetic field $\mathbf{B}_0$. Moreover, each particle carries a spinorial magnetic momentum which is coupled to $\mathbf{B}_0$. In the framework of quantum statistics, the present formalism uses the Feynman-Kac-Itô formula to represent the matrix elements of the quantum Gibbs factor. Particles which are exchanged with one another under a cyclic permutation are equivalent to loops with random shapes; the latter ones obey Maxwell-Boltzmann statistics and interact via some two-body potential which decays as $1/r$ at large distances r. $\mathbf{B}_0$ appears only in a phase factor which can be absorbed in some generalized fugacity (which may take negative values in the case of fermions). Collective Debye screening effects show up through exact systematic resummations of long-ranged Coulomb divergencies which are the same in the presence as in the absence of $\mathbf{B}_0$. The averages of monopole-monopole and monopole-multipole interactions between sets made by charges and their polarization clouds decay exponentially. $\mathbf{B}_0$ breaks the rotational symmetry and effective quantum quadrupolar interactions emerge, as can also be seen in an exactly solvable model. As is also the case for a charge of the medium, an external infinitesimal charge is completely screened by the total charge of the induced polarization cloud. The latter decays as $1/r^5$ as the particle-charge correlation. Subleading tails are also investigated. The interplay with classical Debye screening is discussed. [S1063-651X(98)02610-5]

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I. INTRODUCTION

Matter at our scale can be essentially considered as a nonrelativistic quantum system of electrons and nuclei interacting via the Coulomb potential: the interaction between two point charges e_α and e_γ (where α and γ are species indices) separated by a distance r is $e_\alpha e_\gamma v_C(r)$ with $v_C(r)=1/r$. The standard many-body perturbation theory using Feynman diagrams at finite temperature does not seem to be adequate for tackling the problem of the large-distance behaviors of position correlations [1]; in the special case of the one-component plasma (OCP)—a system made of one species of charges moving in a uniform electric background—one can only exhibit some diagrammatic corrections to the random phase approximation which induce algebraic tails in the charge-charge correlation of the quantum electron gas [2]. Recently path integral formalisms properly adapted to deal with the long range of the Coulomb potential have allowed one to achieve two main results by using methods from statistical mechanics of classical fluids. First, the exact analytical expression for the free energy of these systems has been derived in the low-density regime up to order $\rho^{5/2}$ [3,4] (where ρ is a generic notation for the densities). In the latter references exchange effects were treated perturbatively. Second, a more general formalism [1], which takes quantum statistics systematically into account and where correlations can be studied directly in position space, has been used to exhibit the exponents of algebraic decays for position correlations between quantum charges at large distances [5]. This nonexponential screening, which is contrary to common belief, has been extensively discussed in Refs. [1,5–7].

In the present series of papers (referred to as papers I, II, and III in the following) we give technical details of the derivation of results announced elsewhere. The exact coefficients of the algebraic falloff's of the particle-particle, particle-charge, and charge-charge correlations are derived in the low-density limit first in the case $\mathbf{B}_0=\mathbf{0}$ [8]. This calculation settles the existence of algebraic screening. Moreover, all previous results are revisited in the presence of a uniform magnetic field $\mathbf{B}_0$ [9]. Paper I investigates how the general formalism of Ref. [1] is modified by the presence of $\mathbf{B}_0$ and the new exponents of the algebraic tails of correlations are given. Since the presence of the magnetic field only renormalizes a generalized fugacity in our formalism, low-density expansions can be devised following the same scheme whether $\mathbf{B}_0=\mathbf{0}$ or $\mathbf{B}_0\neq\mathbf{0}$. This is done in Paper II for the exact low-density free energy. (The method is different from that of Ref. [4] and allows one to retrieve the same results in the absence of $\mathbf{B}_0$.) In Paper III the low-density coefficients of the algebraic decays of correlations are derived in the presence as well as in the absence of $\mathbf{B}_0$. We also point out that, when $\mathbf{B}_0\neq\mathbf{0}$, the exact analytical coefficient of the leading algebraic tail for a one-component plasma can be inferred from an exact sum rule specific to the OCP.

In Paper I we argue that even in the presence of $\mathbf{B}_0$, at finite density, monopole-monopole and monople-multipole interactions between charges surrounded by their polarization cloud are exponentially screened at the classical as well

as at the quantum level (see Ref. [1]). When $\mathbf{B}_0=\mathbf{0}$, the large-distance decays of correlations are controlled by ''squared'' quantum fluctuations of some dipolarlike interactions [5], and $\rho_{\alpha\gamma}^{(2)T}(r)|_{\mathbf{B}_0=\mathbf{0}}\sim A_{\alpha\gamma}/r^6$ at large distances r. On the contrary, when $\mathbf{B}_0\neq\mathbf{0}$, the invariance under rotations is broken in one space direction so that quadrupole-quadrupole interactions partially survive after statistical averaging and $\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})|_{\mathbf{B}_0\neq\mathbf{0}}\sim D_{\alpha\gamma}(\hat{\mathbf{r}})/r^5$ when r goes to infinity ($\hat{\mathbf{r}}\equiv\mathbf{r}/|\mathbf{r}|$). (In the absence of rotational invariance, the harmonicity of the Coulomb potential cannot reduce the mean value of quadrupole-quadrupole interactions to short-ranged contributions.)

In the absence of $\mathbf{B}_0$, at any density, the particle-charge and charge-charge correlations, $\Sigma_\gamma e_\gamma\rho_{\alpha\gamma}^{(2)T}(r)|_{\mathbf{B}_0=\mathbf{0}}$ and $\Sigma_{\alpha,\gamma}e_\alpha e_\gamma\rho_{\alpha\gamma}^{(2)T}(r)|_{\mathbf{B}_0=\mathbf{0}}$, fall off as B_α/r^8 and C/r^{10}, respectively, because of the rotational invariance of the problem combined with the harmonicity of the Coulomb potential and some interplay with the partially exponential screening created by other quantum charges (see Ref. [5]). On the contrary, in the presence of $\mathbf{B}_0$, at finite density, the Fourier transforms of correlations involve nonanalytic terms which arise from the breaking of rotational invariance in one space direction and which are not canceled by the harmonicity of the Coulomb potential or by its Debye screening; then the interplay with partially exponential screening does not bring any cascade of inverse power laws for the leading algebraic tails at any density. Even when charges are summed over, all correlations decay as $1/r^5$.

Algebraic screening at large distances is compatible with integral constraints enforced by both internal and perfect external screening, which must also be satisfied in any (classical or quantum) regime. Internal screening refers to the fact that the system, formed by a charge of the medium and its polarization cloud, carries neither any net charge nor any net dipole (see Sec. V B of Ref. [10]), namely,

$$\int d\mathbf{r}\sum_\alpha e_\alpha S_{\alpha\gamma}(\mathbf{r})=0 \tag{1}$$

and

$$\int d\mathbf{r}\,\mathbf{r}\sum_\alpha e_\alpha S_{\alpha\gamma}(\mathbf{r})=\mathbf{0}. \tag{2}$$

In Eq. (2) $S_{\alpha\gamma}(\mathbf{r})$ is the structure factor, $S_{\alpha\gamma}(\mathbf{r})\equiv\rho_\alpha\delta_{\alpha,\gamma}\delta(\mathbf{r})+\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, where $\delta(\mathbf{r})$ is the Dirac distribution and $\delta_{\alpha,\gamma}$ is the Kronecker symbol. Perfect external screening means that the total charge induced in the plasma by an external distribution of charge $\delta q(\mathbf{r})$ exactly compensates the total charge $\int d\mathbf{r}\,\delta q(\mathbf{r})$ in its vicinity. In Fourier space, the property reads

$$\sum_\gamma e_\gamma\rho_\gamma^{\mathrm{ind}}(\mathbf{k}=0)=-\delta q(\mathbf{k}=0). \tag{3}$$

The present paper is organized as follows. The system is defined in Sec. II. In Sec. III we sketch the derivation of the general formalism in the presence of $\mathbf{B}_0$. We recall that, in any representation of many-body states by tensorial products of one-particle states, quantum statistics can be described in terms of cyclic permutations; the general formula for the pressure is checked in the solvable case of a gas of independent charges submitted to $\mathbf{B}_0$ (Sec. III A). For the quantum Gibbs factor in position space we introduce the Feynman-Kac-Itô formula, where $\mathbf{B}_0$ appears only in a phase factor (Sec. III B). The quantum gas of point particles proves to be equivalent to a fluid of ''loops'' with random shapes that obey classical dynamics and Maxwell-Boltzmann statistics and interact via some two-body potential that behaves as $1/r$ at large distances (Sec. III C). Thus, generalized Mayer diagrams may be used. The exact resummation scheme required by Coulomb divergencies at large distances is summed up in Sec. IV. The integrable resummed bonds are listed (Sec. IV A). Those corresponding to monopole-monopole and monopole-multipole loop interactions fall off exponentially over a length scale which tends to the classical Debye value in regimes where exchange effects become negligible. The third one, which describes quantum bound or diffusive states, also involves multipole-multipole loop interactions which generate tails that decay at least as $1/r^3$. A useful diagrammatic representation of the loop density is exhibited: it is equal to the loop fugacity times a function which arises from interactions and involves Mayer diagrams with weight equal to the loop density (Sec. IV B). This integral equation will be useful in the derivation of low-density expansions in Paper II. In Sec. V we present a solvable model in order to exhibit the mechanisms at stake in the presence of the magnetic field. The model consists of two quantum charges embedded in a classical plasma (Sec. V A). It is handled with use of the Feynamn-Kac-Itô formula. When thermal averages are taken for the classical plasma, $\mathbf{B}_0$ disappears from the quantities relative to the classical particles in agreement with the Bohr–van Leeuwen theorem (Sec. V B). The symmetry properties of the covariance of the motion of quantum particles in the classical plasma at finite temperature in the presence of $\mathbf{B}_0$ are studied (Sec. V C). These properties imply that there exists an effective quadrupolar interaction between the two quantum charges. In Sec. VI the leading algebraic tails of static correlations at any density are investigated by an analysis similar to that of Ref. [5]. In Sec. VI A auxiliary bonds are introduced in order to produce an equation à la Dyson which involves convolutions of algebraic tails with functions which decay at least as $1/r^6$ by construction (because their large-distance behaviors necessarily involve some kinds of products of at least two resummed bonds). The intermediate results in the discussion of Ref. [5] that are induced by the invariance under inversion are unchanged (Sec. VI B), whereas the analyticity of some contributions that is enforced by rotational invariance arguments disappears when $\mathbf{B}_0$ is switched on (Sec. VI C). The latter nonanalytic terms are canceled again when the rotational invariance is restored by an integration of the correlation over the angle between $\mathbf{B}_0$ and the relative position of the two particles considered. The study of the leading and subleading behaviors of diagrams is performed in Sec. VII. The algebraic tails before integration over loop shapes have fixed parities under (separate or simultaneous) inversion of loop shapes and their exponents depend on these parities (Sec. VII A). Decays of various kinds of diagrams that fall off at least as $1/r^6$ even before loop-shape integration are discussed in Sec. VII B. Intermediate results are investigated in Appendix A. In Sec.

VII C this study allows one to determine tails of convolutions introduced in Sec. VI A. This survey allows one to derive the leading and subleading algebraic tails of various correlations in Sec. VIII and to check that basic screening rules are satisfied. In Sec. VIII A, we reorganize diagrams in order to use the fact that the ''Debye'' effective monopole-monopole interaction satisfies both the internal and perfect external screening. This allows one to exhibit all algebraic tails of $\rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$, and $\Sigma_{\alpha,\gamma} e_\alpha e_\gamma \rho_{\alpha\gamma}^{(2)T}(\mathbf{r})$. Simultaneously, in Sec. VIII B, we select the diagrams that contribute to the leading asymptotic behaviors which will be calculated at low density in Paper III. In Sec. VIII C, we show that the charge induced by either an internal charge or an infinitesimal external charge is exactly opposite to it and that the density of the induced polarization cloud decays with the same inverse power law as the particle-charge correlation for particles in the plasma. The diagrammatic structure of the leading tail of the induced charge density is also given. Appendixes B and C contain errata for Ref. [5].

II. DEFINITION OF THE SYSTEM

In the present series of papers we consider a multicomponent plasma made of n_s species with index α. Each species is characterized by its mass m_α, its spin $\hbar \mathbf{S}_\alpha$, its charge e_α, and its magnetic momentum $\boldsymbol{\mu}_\alpha = g_\alpha \mu_{B\alpha} \mathbf{S}_\alpha$. $\mu_{B\alpha} = e_\alpha \hbar / 2 m_\alpha c$ is the Bohr magneton and g_α is the Landé factor. The squared spin $\hbar^2 \mathbf{S}_\alpha^2$ takes the values $\hbar^2 S_\alpha (S_\alpha + 1)$, while its component along the z axis, $\hbar [\mathbf{S}_\alpha]_z$, is equal to $\hbar M_\alpha$, with $M_\alpha = -S_\alpha, -S_\alpha + 1, \ldots, S_\alpha$. The dynamical variables of a particle with index i are its position $\mathbf{r}_i$, with conjugate momentum $\mathbf{p}_i = (\hbar / i) \boldsymbol{\nabla}_{\mathbf{r}_i}$, and its spin $\hbar \mathbf{S}_i$. ($\boldsymbol{\nabla}_{\mathbf{r}_i}$ denotes the gradient with respect to the position $\mathbf{r}_i$ and i is the purely imaginary complex number.) In the presence of a uniform magnetic field $\mathbf{B}_0$, we write the Hamiltonian of the system in the nonrelativistic limit as

$$H_{\{N_\alpha\}}(\mathbf{B}_0) = \sum_i \frac{1}{2 m_{\alpha_i}} \left(\mathbf{p}_i - \frac{e_{\alpha_i}}{2c} \mathbf{B}_0 \wedge \mathbf{r}_i \right)^2 - \sum_i g_{\alpha_i} \mu_{B\alpha_i} \mathbf{S}_i \cdot \mathbf{B}_0 + \frac{1}{2} \sum_{i \neq j} e_{\alpha_i} e_{\alpha_j} v_C(\mathbf{r}_i - \mathbf{r}_j), \tag{4}$$

where c is the light velocity, $\wedge$ denotes the outer product, and v_c is defined in Sec. I. $H_{\{N_\alpha\}}(\boldsymbol{B}_0)$ has the following important property. It is the sum of two contributions: one involves only position variables and the other one depends only on spin variables.

The sum of the first two terms in Eq. (4) is the Pauli Hamiltonian for an ideal gas. At thermal equilibrium characterized by a set of densities $\{\rho_\alpha\}_{\alpha = 1, \ldots, n_s}$ and the inverse temperature $\beta = 1/k_B T$, where k_B is the Boltzmann constant and T is the temperature, the corresponding system is stable with Boltzmann statistics. Its thermodynamics involves the two dimensionless parameters $u_{C\alpha} = \beta \mu_{B\alpha} B_0$ and $u_{S\alpha} = (g_\alpha / 2) \beta \mu_{B\alpha} B_0$. These parameters are equal to $\beta/2$ times the cyclotronic energy of orbital motion, $\hbar \omega_{C\alpha}$, and the Larmor energy of spin precession, $\hbar \omega_{L\alpha}$, respectively (with $\omega_{C\alpha} = e_\alpha B_0 / m_\alpha c$ and $\omega_{L\alpha} = g_\alpha \omega_{C\alpha} / 2$).

In the presence of Coulomb interaction, the quantum system is stable only if quantum statistics is taken into account and if all negative and/or positive charges are fermions [11]. The results about the stability of matter in a uniform magnetic field are summarized in Ref. [12]. They deal with the system made of moving electrons and nuclei lying at fixed locations. If the electron spin-field interaction is not included, all proofs of the stability of matter hold with constants unchanged by the substitution of $\mathbf{p}_i$ by $\mathbf{p}_i - (e_{\alpha_i}/c) \mathbf{A}(\mathbf{r}_i)$, where $\mathbf{A}$ is the potential vector. When the spin is taken into account, the contribution from the Zeeman energy $-\Sigma_i \boldsymbol{\mu}_i \cdot \mathbf{B}_0$, which is not bounded below for any arbitrary value of $|\mathbf{B}_0|$, is compensated by the self-energy of the magnetic field $\mathbf{B}_0$ only if $Z\alpha^{*2}$ and α^* are sufficiently small [13]. Ze is the nuclear charge and α^* is the fine structure constant. (Indeed, $Z\alpha^{*2}$ must be small enough to avoid the collapse of an atom, and α^* must be small enough for the repulsion between nuclei to prevent the collapse of a macroscopic number of nuclei.)

The stability also requires that the local neutrality relation

$$\sum_\alpha e_\alpha \rho_\alpha(\mathbf{r}) = 0 \tag{5}$$

might be realized in the bulk. Moreover, even in the presence of $\mathbf{B}_0$, an infinitesimal external charge must be perfectly screened. In the OCP model, the response function does satisfy the corresponding sum rule [See (5.64) in Ref. [10]].

III. GENERAL LOOP FORMALISM

In this section we recall the general formalism of Ref. [1] and we stress the changes that arise in the presence of a uniform magnetic field. This formalism is valid for any quantum system with two-body interaction and quantum statistics.

Let us consider the quantum grand partition function of the system at the inverse temperature β, when a chemical potential μ_α is associated with each species α,

$$\Xi(\beta, \{\mu_\alpha\}, \boldsymbol{B}_0) = \sum_{\{N_\alpha\}_{\alpha = 1, \ldots, n_s}} \mathrm{Tr} \left(e^{-\beta [H_{\{N_\alpha\}}(\boldsymbol{B}_0) - \sum_\alpha \mu_\alpha N_\alpha]} \right). \tag{6}$$

In Eq. (6) the number N_α of particles of species α runs from 0 to ∞. The trace Tr is calculated over a basis of states that are symmetric (antisymmetric) under permutations of particles of each species α according to the bosonic (fermionic) nature of the species α. Moreover, we assume that the thermodynamic limit exists and we consider states in which each particle position may occupy an infinite three-dimensional space. The neutrality relation (5) implies a degeneracy of chemical potentials μ_α [11]; in the thermodynamic limit physical quantities depend only on $n_s - 1$ independent chemical potentials.

A. Quantum statistics and cyclic permutations

In any basis made of tensorial products of one-particle states, the trace Tr, which is a sum over adequately symmetrized/antisymmetrized many-particle states, appears as a sum over permutations π and this sum can be reduced to a sum over cyclic permutations. Indeed, every π can be written as a composition of permutations π_α, each of which involves only one species α of particles, and every π_α itself can be decomposed uniquely as a composition of cycles. Thus a permutation π determines a sequence $\{n_{\alpha,p}\}_{p=1,\ldots,\infty}^{\alpha=1,\ldots,n_s}$ where $n_{\alpha,p}$ is the number of cycles involving p particles of species α in the cyclic decomposition of π_α. The total number of particles of species α can be written as $N_\alpha = \Sigma_p p n_{\alpha,p}$. The decomposition into cycles and the invariance of the Hamiltonian under permutations of particles lead to the expression (A7) of Ref. [1] for Ξ. The point is that in Eq. (A7) the summation may be performed over cycles from the start and the N_α's disappear.

The Hamiltonian (4) does not mix position and spin variables. Therefore, by using the representation of the trace in the particular basis $|\{\mathbf{r}_i, M_i\}\rangle \equiv \otimes_i [|\mathbf{r}_i\rangle \otimes |M_i\rangle]$ (where $\otimes$ denotes a tensorial product), the contributions from the position and spin parts of the Hamiltonian factorize, as in Eq. (A8) of Ref. [1]. [We notice that in Eq. (A8) a $\Pi_{l=1}^p$ is missing in front of the spinorial density-matrix element.] Moreover, since the Zeeman term of the Hamiltonian is diagonal in the basis $\otimes_i |M_i\rangle$, this factorization implies that the only configurations of spin states that give nonvanishing contributions are those in which all particles of species α involved in the same cycle are in the same spin state M_α.

Eventually, a notion of loop can be associated with each cyclic permutation of positions as follows. When the spin configurations are summed over independently from the position configurations, Ξ is given by Eq. (3.1) of Ref. [1] with the following change: for each cycle with p particles of species α, the spin degeneracy factor $2S_\alpha + 1$ is replaced by

$$\sum_{M_\alpha=-S_\alpha}^{S_\alpha} (\exp[\beta g_\alpha \mu_{B\alpha} M_\alpha B_0])^p = \sinh([2S_\alpha+1]pu_{S\alpha})/\sinh(pu_{S\alpha}).$$

The result is

$$\Xi = \sum_{\{n_{\alpha,p}\}_{p=1,\ldots,\infty}^{\alpha=1,\ldots,n_s}} \prod_{n_{\alpha,p}\neq 0} \frac{1}{n_{\alpha,p}!} \times \left(\frac{\eta_\alpha^{p-1}}{p} \frac{\sinh([2S_\alpha+1]pu_{S\alpha})}{\sinh(pu_{S\alpha})} e^{p\beta\mu_\alpha} \right)^{n_{\alpha,p}} \times \int \prod_i d\mathbf{r}_i \langle \{\mathbf{r}_{\pi(i)}\} | e^{-\beta H_{\{n_{\alpha,p}\}}(\mathbf{B}_0)} | \{\mathbf{r}_i\} \rangle, \quad (7)$$

where $|\{\mathbf{r}_i\}\rangle \equiv \otimes_i |\mathbf{r}_i\rangle$ and π is a particular composition of permutations π_α corresponding to the sequence $\{n_{\alpha,p}\}$ and i ranges from 1 to $\Sigma_\alpha \Sigma_p p n_{\alpha,p}$. $\eta_\alpha \equiv (-1)^{2S_\alpha}$ is equal to 1 for bosons and to -1 for fermions. It arises from the signature of the permutation π_α (which is equal to η_α to the power $N_\alpha - \Sigma_{p=1}^{N_\alpha} n_{\alpha,p}$). The symmetry factor $1/p$ comes from the arbitrariness in the choice of the particle that is labeled with number 1 among the p particles involved in the cycle. $[\exp(\beta\mu_\alpha)]^p$ is the dimensionless fugacity associated with p particles. The loop denoted by $\mathcal{L}^*$ is the set of degrees of freedom $(\alpha, p, \{\mathbf{x}_1, \ldots, \mathbf{x}_p\})$, where the positions are labeled according to the order of their transformation in the corresponding cyclic permutation, i.e., $\pi_\alpha(\mathbf{x}_i) = \mathbf{x}_{i+1}$, with the convention $\mathbf{x}_{p+1} \equiv \mathbf{x}_1$. With the notation $\int d\mathcal{L}^* \cdots \equiv \Sigma_{\alpha=1}^{n_s} \Sigma_{p=1}^{\infty} \int \Pi_{l=1}^p d\mathbf{x}_l \cdots$, the summation over cycles can be written as a sum over loops, with

$$\sum_{\{n_{\alpha,p}\}_{p=1,\ldots,\infty}^{\alpha=1,\ldots,n_s}} \prod_{n_{\alpha,p}\neq 0} \frac{1}{n_{\alpha,p}!} \int \prod_i d\mathbf{r}_i \cdots = \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{n=1}^{N} d\mathcal{L}_n^* \cdots. \quad (8)$$

If the spin state M_α is kept as an extra internal degree of freedom of the loop, then $\mathcal{L}^*$ is replaced by $\tilde{\mathcal{L}}^* = (\alpha, M_\alpha, p, \{\mathbf{x}_1, \ldots, \mathbf{x}_p\})$. The identity (8) is still valid with n_{α,p,M_α} in place of $n_{\alpha,p}$, while in Eq. (7) the sum over the values M_α of $[\mathbf{S}]_z$, $\sinh([2S_\alpha+1]pu_{S\alpha})/\sinh(pu_{S\alpha})$, is replaced by a single term $(\exp[\beta g_\alpha \mu_{B\alpha} M_\alpha B_0])^p$. The latter representation is the most adequate one for the following case.

We can check that the present formalism allows us to retrieve the pressure of an ideal gas with quantum statistics in a uniform magnetic field $\mathbf{B}_0$. The grand partition function of the latter system may be written as

$$\Xi_0 = \sum_{N=0}^{\infty} \frac{1}{N!} \int \left[\prod_{n=1}^{N} d\tilde{\mathcal{L}}_n^* \right] \prod_{n=1}^{N} z_0(\tilde{\mathcal{L}}_n^*) = \exp\left[\int d\tilde{\mathcal{L}}^* z_0(\tilde{\mathcal{L}}^*) \right], \quad (9)$$

with $\int d\tilde{\mathcal{L}}^* \cdots \equiv \Sigma_{\alpha=1}^{n_s} \Sigma_{M_\alpha=-S_\alpha}^{S_\alpha} \Sigma_{p=1}^{\infty} \int \Pi_{l=1}^p d\mathbf{x}_l \cdots$ and the ''fugacity''

$$z_0(\tilde{\mathcal{L}}^*) \equiv \frac{\eta_\alpha^{p-1}}{p} (e^{\beta[\mu_\alpha + g_\alpha \mu_{B\alpha} B_0 M_\alpha]})^p \times \prod_{l=1}^{p} \langle \mathbf{x}_1 | e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}} | \mathbf{x}_p \rangle \times \langle \mathbf{x}_p | e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}} | \mathbf{x}_{p-1} \rangle \cdots \langle \mathbf{x}_2 | e^{-\beta h_{\mathbf{B}_0,\alpha}^{(0)}} | \mathbf{x}_1 \rangle. \quad (10)$$

In Eq. (10) the chemical potential μ_α is shifted by the Zeeman energy $g_\alpha \mu_{B\alpha} B_0 M_\alpha$, which removes the degeneracy between the spin states. $h_{\mathbf{B}_0,\alpha}^{(0)}$ is the one-body Hamiltonian of a particle without spin in the magnetic field,

$$h_{\mathbf{B}_0,\alpha}^{(0)} = 1/(2m_\alpha)[\mathbf{p} - (e_\alpha/2c)\mathbf{B}_0 \wedge \mathbf{x}]^2.$$

After integration over the positions $\mathbf{x}_l$, the product of matrix elements in Eq. (10) gives a factor $\mathrm{Tr}(\exp[-p\beta h_{\mathbf{B}_0,\alpha}^{(0)}])$. Let us choose the magnetic field along the z axis, $\mathbf{B}_0 = B_0 \mathbf{e}_z$, where $\mathbf{e}_z$ is a unit vector. The Landau energy levels of $h_{\mathbf{B}_0,\alpha}^{(0)}$ are

$$\varepsilon_\alpha^{(0)}([\mathbf{k}]_z,n)=\frac{\hbar^2[\mathbf{k}]_z^2}{2m_\alpha}+\hbar\omega_{C\alpha}\left(n+\frac{1}{2}\right), \tag{11}$$

where $[\mathbf{k}]_z$ is the component of $\mathbf{k}$ along the z axis and n is a positive integer, $n=0,1,2,\ldots$. For $[\mathbf{k}]_z$ and n fixed, the degeneracy factor in a box with volume Λ is $\Lambda^{1/2}/(2\pi)$ times $\Lambda^{2/3}e_\alpha B_0/2\pi\hbar c$ [14] and

$$\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\mathrm{Tr}(e^{-p\beta h^{(0)}_{\mathbf{B}_0,\alpha}})=\frac{e_\alpha B_0}{2\pi\hbar c}\int\frac{d[\mathbf{k}]_z}{2\pi}\sum_{n=0}^{\infty}[e^{-\beta\varepsilon_\alpha^{(0)}([\mathbf{k}]_z,n)}]^p. \tag{12}$$

The summation over p can be performed according to the identity $\ln(1-x)=-\Sigma_{p=1}^{\infty}x^p/p$ (In principle, the latter identity is valid only when $-1\leqslant x<1$, but it can be used for any x by analytical continuation.) Finally, we retrieve the pressure $P^{(0)}$ of a gas of independent quantum charges in a uniform magnetic field [14],

$$\begin{aligned}\beta P^{(0)}&=\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\ln\Xi_0(\beta,\{\mu_\alpha\},B_0;\Lambda)\\&=\sum_{\alpha=1}^{n_s}\sum_{M_\alpha=-S_\alpha}^{S_\alpha}\frac{e_\alpha B_0}{2\pi\hbar c}\int\frac{d[\mathbf{k}]_z}{2\pi}\sum_{n=0}^{\infty}(-\eta_\alpha)\\&\quad\times\ln[1-\eta_\alpha e^{\beta\{\mu_\alpha+g_\alpha\mu_{B\alpha}B_0M_\alpha-\varepsilon_\alpha^{(0)}([\mathbf{k}]_z,n)\}}].\end{aligned} \tag{13}$$

B. Feynman-Kac-Itô path integral

The representation of the quantum Gibbs factor in terms of noncommuting operators is replaced in Eq. (7) by a representation in terms of scalar functional integrals, by using the Feynman-Kac formula. In the Feynman-Kac path integral the presence of $\mathbf{B}_0$ only introduces an extra phase factor $\exp[(ie_\alpha/\hbar c)\int_{\mathrm{FKac}}\mathbf{A}\cdot d\boldsymbol{\omega}]$ where $d\boldsymbol{\omega}$ is a line element of the path $\boldsymbol{\omega}$ [15]. $\int_{\mathrm{FKac}}\mathbf{A}\cdot d\boldsymbol{\omega}$ is defined as the limit of either the discrete sum of terms $(\boldsymbol{\omega}_n-\boldsymbol{\omega}_{n-1})\cdot\mathbf{A}([\boldsymbol{\omega}_n+\boldsymbol{\omega}_{n-1}]/2)$ or the sum of $(\boldsymbol{\omega}_n-\boldsymbol{\omega}_{n-1})\cdot[\mathbf{A}(\boldsymbol{\omega}_n)+\mathbf{A}(\boldsymbol{\omega}_{n-1})]/2$ when the discrete dimensionless ''time'' spacing Δt between $\boldsymbol{\omega}_n$ and $\boldsymbol{\omega}_{n-1}$ goes to zero with the scaling law of a Brownian walk,

$$[\boldsymbol{\omega}_n-\boldsymbol{\omega}_{n-1}]_\mu[\boldsymbol{\omega}_n-\boldsymbol{\omega}_{n-1}]_\nu\underset{\Delta t\to0}{\sim}\delta_{\mu,\nu}\lambda_\alpha^2\Delta t. \tag{14}$$

The Schrödinger equation may be derived by writing the difference between the wave functions at times t and $t+\Delta t$ infinitesimally close together and by using the fact that the quantum Gibbs factor for one particle is the kernel of the integral representation of the evolution of the wave function in imaginary time. The use of the discrete sums defined above ensures that the latter Schrödinger equation coincides with that obtained from the usual quantization of the classical Hamiltonian.

In fact, from the mathematical point of view, the Itô integral $\int_{\mathrm{Itô}}\mathbf{A}\cdot d\boldsymbol{\omega}$ must be used in order to properly define the integral $\int\mathbf{A}\cdot d\boldsymbol{\omega}$ in functionals involving averages over Brownian paths [16,17]. The Itô integral corresponds to the discrete sum of terms $(\boldsymbol{\omega}_n-\boldsymbol{\omega}_{n-1})\cdot\mathbf{A}(\boldsymbol{\omega}_{n-1})$ where $(\boldsymbol{\omega}_n-\boldsymbol{\omega}_{n-1})$ points towards the future and $\mathbf{A}(\boldsymbol{\omega}_{n-1})$ only depends on the past. Its relation with the Feynman-Kac integral reads

$$\int_{\mathrm{FKac}}\mathbf{A}\cdot d\boldsymbol{\omega}=\int_{\mathrm{Itô}}\mathbf{A}\cdot d\boldsymbol{\omega}+\frac{1}{2}\lambda_\alpha^2\int_0^1 ds\,\boldsymbol{\nabla}\cdot\mathbf{A}(\boldsymbol{\omega}(s)), \tag{15}$$

where s is a dimensionless abscissa ($ds\equiv-i\Delta t$).

If $\boldsymbol{\omega}$ is a closed path, $\int_{\mathrm{FKac}}\mathbf{A}\cdot d\boldsymbol{\omega}$ is gauge invariant. Indeed, in a gauge transformation $\mathbf{A}\to\mathbf{A}+\boldsymbol{\nabla}f$, and $U\to U-\partial f/\partial t$, where U is the electromagnetic scalar potential, $\int_0^1 ds\,U(s)+\int_{\mathrm{FKac}}\mathbf{A}\cdot d\boldsymbol{\omega}$ is just translated by $f(\boldsymbol{\omega}(s=1))-f(\boldsymbol{\omega}(s=0))$, as it should be. This transformation property is ensured by the Itô lemma for a function $f(\boldsymbol{\omega}(s),s)$,

$$\begin{aligned}&\int_{\mathrm{Itô}}\boldsymbol{\nabla}f(\boldsymbol{\omega}(s),s)\cdot d\boldsymbol{\omega}+\frac{1}{2}\lambda_\alpha^2\int_0^1 ds\,\Delta f(\boldsymbol{\omega}(s),s)\\&\quad+\int_0^1 ds\,\frac{\partial f(\boldsymbol{\omega}(s),s)}{\partial s}\\&\quad=f(\boldsymbol{\omega}(s=1),s=1)-f(\boldsymbol{\omega}(s=0),s=0).\end{aligned} \tag{16}$$

If $\mathbf{B}_0$ is uniform, one can choose the Coulomb gauge where $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. Then, according to Eq. (15), $\int_{\mathrm{FKac}}\mathbf{A}\cdot d\boldsymbol{\omega}=\int_{\mathrm{Itô}}\mathbf{A}\cdot d\boldsymbol{\omega}$, and the subscript Itô will be omitted.

In the following, we choose the Coulomb gauge which is isotropic in the plane perpendicular to $\mathbf{B}_0$, namely, $\mathbf{A}(\mathbf{r})=(1/2)\boldsymbol{B}_0\wedge\mathbf{r}$. In this gauge, the Feynman-Kac-Itô formula reads

$$\begin{aligned}&\langle\{\mathbf{r}_{\pi(i)}\}|e^{-\beta H_{\{n_{\alpha,p}\}}}|\{\mathbf{r}_i\}\rangle\\&=\prod_i\left[\frac{1}{(2\pi\lambda_{\alpha_i}^2)^{3/2}}e^{-(\mathbf{r}_{\pi(i)}-\mathbf{r}_i)^2/2\lambda_{\alpha_i}^2}\right]\\&\quad\times\int\left[\prod_i D(\boldsymbol{\xi}_i)\right]\exp\left[\sum_i(ie_{\alpha_i}/2\hbar c)\right.\\&\quad\left.\times\mathbf{B}_0\cdot\int_0^1\boldsymbol{\omega}_{i,\pi(i)}(s)\wedge d\boldsymbol{\omega}_{i,\pi(i)}\right]\\&\quad\times\exp\left[-\beta(1/2)\Sigma_{i\neq j}e_{\alpha_i}e_{\alpha_j}\right.\\&\quad\left.\times\int_0^1 ds\,v_C(\boldsymbol{\omega}_{i,\pi(i)}(s)\quad-\boldsymbol{\omega}_{j,\pi(j)}(s))\right].\end{aligned} \tag{17}$$

$\boldsymbol{\omega}_{i,\pi(i)}(s)$ is a Brownian path starting from $\mathbf{r}_i$ at $s=0$ and ending at $\mathbf{r}_{\pi(i)}$ at $s=1$. It can be decomposed into a uniform motion along a straight line linking $\mathbf{r}_i$ to $\mathbf{r}_{\pi(i)}$ plus a random fluctuation,

$$\boldsymbol{\omega}_{i,\pi(i)}(s)=(1-s)\,\mathbf{r}_i+s\,\mathbf{r}_{\pi(i)}+\lambda_{\alpha_i}\boldsymbol{\xi}_i(s), \tag{18}$$

where λ_{α_i} is the thermal de Broglie wavelength defined as $\lambda_{\alpha_i}=(\beta\hbar^2/m_{\alpha_i})^{1/2}$, and $\boldsymbol{\xi}_i(s)$ is a dimensionless Brownian bridge which vanishes when $s=0$ and $s=1$. The measure $D(\boldsymbol{\xi})$, which contains the exponential of the kinetic part of the Euclidean action, is normalized, $\int D(\boldsymbol{\xi})=1$, and has a Gaussian covariance, which is independent from the species,

$$\mathrm{cov}_{\mu\nu}(s,s';\mathbf{B}_0=\mathbf{0})\equiv\int D(\boldsymbol{\xi})[\boldsymbol{\xi}(s)]_\mu[\boldsymbol{\xi}(s')]_\nu = \delta_{\mu,\nu}\mathrm{inf}(s,s')[1-\mathrm{sup}(s,s')], \tag{19}$$

where $\mathrm{inf}(s,s')[\mathrm{sup}(s,s')]$ denotes the infimum (supremum) of s and s'.

C. Equivalence with a Maxwell-Boltzman gas of classical loops

As a consequence of the two previous sections, and as in the absence of the magnetic field, a particle that is not exchanged under any cyclic permutation is associated with a closed path $\boldsymbol{\omega}_{1,1}$, whereas p particles that are involved in a given cyclic permutation are described by p open Brownian paths $\boldsymbol{\omega}_{l,l+1}$, with $l=1,\ldots,p$. ($\boldsymbol{\omega}_{l,l+1}$ links $\mathbf{x}_l$ to $\mathbf{x}_{l+1}$ with the convention $\mathbf{x}_{p+1}=\mathbf{x}_1$.) These open paths form a closed curve $\boldsymbol{\Omega}$ which is parametrized by an abscissa τ ranging from 0 to p,

$$\boldsymbol{\Omega}(\tau)=\sum_{l=1}^{p}\delta_{P(\tau),l-1}\boldsymbol{\omega}_{l,l+1}[\tau-(l-1)]. \tag{20}$$

In Eq. (20) $P(\tau)$ denotes the integral part of τ. For instance, $\boldsymbol{\Omega}(\tau=0)\equiv\boldsymbol{\omega}_{1,2}(s=0)=\mathbf{x}_1$, and we set $\boldsymbol{\Omega}(\tau=p)\equiv\boldsymbol{\omega}_{p,1}(s=1)=\mathbf{x}_1$. In the following, we call $\mathbf{R}\equiv\mathbf{x}_1$ the ''position'' of the loop and $\mathbf{X}(\tau)\equiv\boldsymbol{\Omega}(\tau)-\mathbf{R}$ its ''shape.'' A loop $\mathcal{L}$ is described by its position and its internal degrees of freedom $(\alpha,p,\mathbf{X})$ (when the spin configurations are summed over). In the following, p will be called the exchange degeneracy of the loop. We define the integration measure $\int d\mathcal{L}=\int d\mathbf{R}\int D(\mathbf{X})$ with

$$D(\mathbf{X})\equiv\prod_{l=2}^{p}d\mathbf{x}_l\prod_{l=1}^{p}D(\boldsymbol{\xi}_l). \tag{21}$$

[We notice that λ_α, which is involved in the definition of $\mathbf{X}$, does not appear in the measure $D(\mathbf{X})$.] With these notations, according to Eqs. (7) and (17), the grand partition function (6) of a system of quantum particles with quantum statistics and interacting via a two-body potential and with a uniform external magnetic field $\mathbf{B}_0$ according to Eq. (4) can be written as the grand partition function of a system of classical loops with Maxwell-Boltzmann (MB) statistics and interacting via some two-body potential, as in Eq. (3.11) of Ref. [1],

$$\Xi=\Xi_{\mathrm{loop}}=\sum_{N=O}^{\infty}\frac{1}{N!}\int\prod_{n=1}^{N}[z(\mathcal{L}_n)d\mathcal{L}_n]e^{-\beta(1/2)\sum_{i\neq j}e_{\alpha_i}e_{\alpha_j}v(\mathcal{L}_i,\mathcal{L}_j)}. \tag{22}$$

In Eq. (22) we use the convention that, if $N=0$, there is no $\mathcal{L}_n$ in the corresponding term of Ξ_{loop} and the latter term is merely equal to 1. The potential between loops can be expressed as

$$v(\mathcal{L}_i,\mathcal{L}_j)=\int_0^{p_i}d\tau\int_0^{p_j}d\tau'\ \delta([\tau-P(\tau)]-[\tau'-P(\tau')])\times v_C(\boldsymbol{\Omega}_i(\tau)-\boldsymbol{\Omega}_j(\tau')). \tag{23}$$

The only difference with the case $\mathbf{B}_0=\mathbf{0}$ is an extra phase factor which has been incorporated in the fugacity $z(\mathcal{L})$. The phase factor involves $\int_0^p\boldsymbol{\Omega}(\tau)\wedge d\boldsymbol{\Omega}\equiv\int_0^p d\tau\,\boldsymbol{\Omega}(\tau)\wedge[d\boldsymbol{\Omega}(\tau)/d\tau]$, but the latter reduces to $\int_0^p\mathbf{X}(\tau)\wedge d\mathbf{X}$. Indeed, when $\boldsymbol{\Omega}$ is changed into $\boldsymbol{\Omega}+\mathbf{a}$, $\int_0^p\boldsymbol{\Omega}(\tau)\wedge d\boldsymbol{\Omega}$ is translated by $\mathbf{a}\wedge\int_0^p d\boldsymbol{\Omega}(\tau)=\mathbf{a}\wedge\sum_{l=1}^p[(\mathbf{x}_{l+1}-\mathbf{x}_l)+\lambda_\alpha\int_0^1 d\boldsymbol{\xi}_l(s)]$ which vanishes because $\mathbf{x}_{p+1}=\mathbf{x}_1$ and $\int_0^1 d\boldsymbol{\xi}_l(s)=\mathbf{0}$, as a consequence of Itô lemma (16) applied to the function $f(\boldsymbol{\omega}(s),s)=\boldsymbol{\omega}(s)\cdot\mathbf{e}$ where $\mathbf{e}$ is a unit vector with any given orientation. With the same notations as in Ref. [1],

$$z(\mathcal{L})=z^*_{\alpha,p}e^{(ie_\alpha/2\hbar c)\mathbf{B}_0\cdot\int_0^p\mathbf{X}(\tau)\wedge d\mathbf{X}(\tau)}e^{-\beta E^{\mathrm{int}}_{\beta,\alpha}(\mathbf{X})}, \tag{24}$$

where $E^{\mathrm{int}}_{\beta,\alpha}$ is an internal energy which does not depend on $\mathbf{B}_0$,

$$E^{\mathrm{int}}_{\beta,\alpha}(\mathbf{X})=\frac{1}{\beta 2\lambda_\alpha^2}\sum_{l=1}^{p}(\mathbf{x}_{l+1}-\mathbf{x}_l)^2+\frac{1}{2}e_\alpha^2\int_0^p d\tau\int_0^p d\tau'\times(1-\delta_{P(\tau),P(\tau')})\,\delta([\tau-P(\tau)]-[\tau'-P(\tau')])\times v_C(\boldsymbol{\Omega}(\tau)-\boldsymbol{\Omega}(\tau')) \tag{25}$$

and

$$z^*_{\alpha,p}=\frac{\eta_\alpha^{p-1}}{p}\frac{\sinh([2S_\alpha+1]pu_{S\alpha})}{\sinh(pu_{S\alpha})}\left(\frac{e^{\beta\mu_\alpha}}{(2\pi\lambda_\alpha^2)^{3/2}}\right)^p. \tag{26}$$

In Eq. (26) η_α^{p-1} is a memory of quantum statistics and the symmetry factor p comes from the arbitrary choice for the loop position $\mathbf{R}$ among the p particle positions involved in the loop. The paramagnetic contribution $\sinh([2S_\alpha+1]pu_{S\alpha})/\sinh(pu_{S\alpha})$ reduces to the spin degeneracy $2S_\alpha+1$ when the magnetic field vanishes, while $e^{\beta\mu_\alpha}/(2\pi\lambda_\alpha^2)^{3/2}$ is the usual dimensioned fugacity. We notice that the Gaussian part arising from $E^{\mathrm{int}}_{\beta,\alpha}(\mathbf{X})$ in Eq. (24) together with the phase factor generated by the coupling with $\mathbf{B}_0$ could be absorbed in the measure $D(\mathbf{X})$ so that $E^{\mathrm{int}}_{\beta,\alpha}(\mathbf{X})$ would reduce to the Coulomb self-energy. However, we do not choose this decomposition, because we want to keep an explicit track of the positions of the various particles involved in a loop in order to study the correlations between the positions of quantum particles in the following.

The important properties of $z(\mathcal{L})$ are the following. $z(\mathcal{L})$ depends only on the shape $\mathbf{X}$ of the loop, and not on its position $\mathbf{R}$, $z(\mathcal{L})\equiv z_{\alpha,p}(\mathbf{X})$. It is unchanged under a gauge transformation because the phase factor due to the presence of $\mathbf{B}_0$ involves $\int\mathbf{A}\cdot d\boldsymbol{\omega}$ calculated on a closed curve. [See the comment just before Eq. (16).] Moreover, $z_{\alpha,p}(\mathbf{X})$ is invariant under the inversion $\mathbf{X}\rightarrow-\mathbf{X}$ and under the rotation of $\mathbf{X}$ around $\mathbf{B}_0$.

The gas of loops obeys Maxwell-Boltzmann statistics so that usual techniques from classical statistical mechanics of fluids can be applied. This was done by Ginibre in order to prove the convergence of low-density expansions of thermo-

dynamic functions for some integrable potentials [18]. The methods were different from those used in the following. In the present paper, as in Ref. [1], we take advantage of the fact that the grand partition function (22) is a functional of the loop fugacity in order to introduce Mayer diagrams. Indeed, since the loop density and the correlations between the loops can be expressed as functional derivatives of the grand partition function, the rules for the Mayer diagrams are the same as for point objects. In these diagrams there is at most one Mayer bond $f \equiv \exp[-\beta v]-1$ between two points and a point, which represents the variables of one loop, is associated with an integration measure $d\mathcal{L} \equiv \int d\mathbf{r} \int D(\mathbf{X})$. By definition, the ''internal'' points are integrated over, while the ''root'' points are not.

We will use a diagrammatic representation of the loop density that was not introduced in Ref. [1], and that has not be used in the literature for fluids of point particles, at least to our knowledge. This representation, which may be interpreted as an integral equation for the loop density, reads

$$\rho(\mathcal{L}_a) = z(\mathcal{L}_a) \exp\left\{ \sum_{\mathbb{G}^*} \frac{1}{S_{\mathbb{G}^*}} \int \prod_{n=1}^{N} [d\mathcal{L}_n \rho(\mathcal{L}_n)] \left[\prod f \right]_{\mathbb{G}^*} \right\}. \tag{27}$$

Equation (27) may be derived from the usual fugacity expansion of the density where the density appears as the sum of all unlabeled topologically different connected diagrams with one root point $\mathcal{L}_a$ and N internal points ($N=1,\ldots,\infty$). In Eq. (27) the sum runs over all diagrams $\mathbb{G}^*$, which satisfy the previous definition with two additional constraints: they contain no articulation point and they remain as a single piece when all bonds involving the root point are cut. The last property must be satisfied because the expansion of the exponential of the sum of such diagrams generates all diagrams and, in particular, all of those that do not remain as a single piece when the root point is removed. Moreover, an articulation point is such that, if the bonds with which it is involved are cut, then the diagram is split into two pieces and at least one of these pieces will no longer be linked to the root point. The absence of the articulation point comes from the fact that each internal point $\mathcal{L}_n$ of the diagram is weighted by the density $\rho(\mathcal{L}_n)$ and not by the fugacity $z(\mathcal{L}_n)$. $S_{\mathbb{G}^*}$ is the symmetry factor of a given graph $\mathbb{G}^*$, namely, the number of permutations of the internal points $\mathcal{L}_n$ that do not change the integrand $[\Pi f]_{\mathbb{G}^*}$, which is the product of all Mayer bonds in $\mathbb{G}^*$. Moreover, it is convenient to write the truncated two-loop distribution function $\rho^{(2)T}(\mathcal{L}_a,\mathcal{L}_b)$ as $\rho^{(2)T}(\mathcal{L}_a,\mathcal{L}_b) = \rho(\mathcal{L}_a)\rho(\mathcal{L}_b)h(\mathcal{L}_a,\mathcal{L}_b)$, where the loop Ursell function $h(\mathcal{L}_a,\mathcal{L}_b)$ can be simply expressed as

$$h(\mathcal{L}_a,\mathcal{L}_b) = \sum_{\Gamma} \frac{1}{S_\Gamma} \int \prod_{n=1}^{N} [d\mathcal{L}_n \rho(\mathcal{L}_n)] \left[\prod f \right]_{\Gamma}. \tag{28}$$

In Eq. (28) the sum runs over all unlabeled topologically different connected diagrams Γ with two root points $\mathcal{L}_a$ and $\mathcal{L}_b$ and N internal points ($N=0,\ldots,\infty$) without any articulation point. The contribution for $N=0$ reduces to $f(\mathcal{L}_a,\mathcal{L}_b)$.

The quantum particle densities and n-body distribution functions are derived from the loop distribution functions by integration over some internal degrees of freedom of the loops. The formulas are the same as in Eqs. (4.3), (4.6), and (4.7) of [1] and they will be recalled when they are used in the following.

As shown below, the Mayer graphs for the loop system are very useful to investigate the large-distance behavior of correlations between quantum particles directly in position space or to devise systematic low-density expansions for thermodynamic quantities or position correlations in terms of the densities of quantum particles.

IV. SPECIAL CASE OF COULOMB INTERACTION

In the case of the long-ranged Coulomb interaction, the Mayer graphs diverge and graphs must be collected in order to get new graphs with integrable resummed bonds. In this section we only summarize the steps of the exact resummation process, because it is a straightforward generalization of the procedure detailed in Ref. [1] in the case $\mathbf{B}_0 = \mathbf{0}$. We first address the resummation scheme for the loop-density expansion of the loop Ursell function, because the topological principles are simpler than for the loop-fugacity expansion of the loop density which will be discussed in Sec. IV B.

A. Exact resummation of Coulomb divergencies for the loop Ursell function

The interaction between two loops may be decomposed into the sum of three kinds of contributions: monopole-monopole and multipole-monopole interactions, which are identical to their electrostatic analogs, and a multipole-multipole interaction, which cannot be interpreted as an electrostatic energy, because the Feynman-Kac formula involves only interactions between loop line elements with the same abscissa (up to an integer). Auxiliary Mayer bonds are introduced according to this decomposition, and the auxiliary diagrams are collected inside equivalence classes in order to sum convolution chains of auxiliary bonds where the intermediate points are so-called Coulomb points. The definition of a Coulomb point is the following: it appears in the auxiliary interaction bonds only through the monopole of the corresponding loop, namely, through its total charge.

The presence of the magnetic field does not modify the resummation process, because the latter involves only the large-distance behavior of the loop interaction, $-\beta v_C(|\mathbf{R}_i - \mathbf{R}_j|)$, which does not depend on the shape of the loops. The whole Sec. V of Ref. [1] is unchanged, except that the value of $\int D(\mathbf{X})\rho(\mathbf{X})$ now depends on $\mathbf{B}_0$. As a result, the Ursell function can be expressed as a sum over Mayer diagrams Π,

$$h(\mathcal{L}_a,\mathcal{L}_b) = \sum_{\Pi} \frac{1}{S_\Pi} \int \prod_{m=1}^{M} [d\mathcal{P}_m \rho(\mathcal{P}_m)] \left[\prod F \right]_{\Pi}. \tag{29}$$

Equation (29) is analogous to Eq. (28) with the only difference that, in order to avoid double counting, there exist three kinds of resummed bonds with a related excluded-convolution rule when the intermediate point is involved only through its total charge. (See next paragraph.)

The resummed bonds F^{cc} and F^{cm} corresponding to the monopole-monopole, i.e., charge-charge, and monopole-multipole, i.e., charge-multipole, interactions decay exponentially. More precisely

$$F^{cc}(\mathcal{L}_i,\mathcal{L}_j) = -\beta p_i e_{\alpha_i} p_j e_{\alpha_j} \phi(|\mathbf{R}_i - \mathbf{R}_j|) \tag{30}$$

while

$$F^{mc}(\mathcal{L}_i,\mathcal{L}_j) = -\beta p_i e_{\alpha_i} p_j e_{\alpha_j} \left[\int_0^{p_i} \frac{d\tau}{p_i} \phi(|\mathbf{\Omega}_i(\tau) - \mathbf{R}_j|) - \phi(|\mathbf{R}_i - \mathbf{R}_j|) \right] \tag{31}$$

and, in a similar way,

$$F^{cm}(\mathcal{L}_i,\mathcal{L}_j) = -\beta p_i e_{\alpha_i} p_j e_{\alpha_j} \left[\int_0^{p_j} \frac{d\tau}{p_j} \phi(|\mathbf{R}_i - \mathbf{\Omega}_j(\tau)|) - \phi(|\mathbf{R}_i - \mathbf{R}_j|) \right]. \tag{32}$$

In these equations the potential is of Debye form,

$$\phi(R) = \frac{\exp(-\kappa R)}{R}. \tag{33}$$

The expression of κ is the same as in Eq. (5.14) of Ref. [1],

$$\kappa = \left[4\pi\beta \sum_{\alpha=1}^{n_s} e_\alpha^2 \sum_{p=1}^{\infty} p^2 \int D(\mathbf{X}) \rho_{\alpha,p}(\mathbf{X}) \right]^{1/2}. \tag{34}$$

In the quantum weak coupling regime for fermions at high density, it tends to its value in the random phase approximation, whereas, in the classical limit where exchange effects become negligible at low density, it coincides with the Debye inverse length κ_D^{-1} with $\kappa_D^2 = 4\pi\beta\Sigma_\alpha e_\alpha^2 \rho_\alpha$. The excluded convolution rule is the following: there can be no convolution $F^{cc} * F^{cc}$, $F^{cc} * F^{cm}$, or $F^{mc} * F^{cm}$, where $*$ denotes a convolution for the loop position variable $\mathbf{R}$ and an integration over the internal degrees of freedom of the intermediate loop.

On the contrary, the dressed bond, which contains the multipole-multipole interactions and the short-ranged part of the Mayer bond f, decays algebraically at large distances. It reads

$$F_R = e^{-\beta e_{\alpha_i} e_{\alpha_j} \phi_{\text{elect}} + W} - 1 - F^{cc} - F^{cm} - F^{mc}, \tag{35}$$

with

$$W(\mathcal{L}_i,\mathcal{L}_j) = -\beta e_{\alpha_i} e_{\alpha_j} \int_0^{p_i} d\tau \int_0^{p_j} d\tau' \{ \delta([\tau - P(\tau)] - [\tau' - P(\tau')]) - 1 \} v_C(\mathbf{\Omega}_i(\tau) - \mathbf{\Omega}_j(\tau')) \tag{36}$$

and

$$\phi_{\text{elect}}(\mathcal{L}_i,\mathcal{L}_j) = \int_0^{p_i} d\tau \int_0^{p_j} d\tau' \; \phi(\mathbf{\Omega}_i(\tau) - \mathbf{\Omega}_j(\tau')). \tag{37}$$

When the distance $|\mathbf{R}_i - \mathbf{R}_j|$ between loops becomes infinite, the leading asymptotic term in W decays as $1/|\mathbf{R}_i - \mathbf{R}_j|^3$. The algebraic tails of the bond F_R are generated by the expansion of $\exp(W) - 1$. We notice that F_R depends on the density only through the inverse length κ.

B. Resummed diagrammatic representation of the loop density

Before using Eq. (27), we must study the representation of the loop density $\rho(\mathcal{L})$ in terms of diagrams with weight $z(\mathcal{L})$. Resummations are the same as in Appendix B of Ref. [1]. Moreover, a generalization of Appendix C shows that each resummed diagram is conditionally integrable, if the integrations over the loop shapes and over the orientations of the relative positions of loops are performed before the integrations over the distances between loops. More precisely, since articulation points exist in diagrams with weights $z(\mathcal{L})$, a point $\mathcal{P}_j$ may be linked to only one point $\mathcal{P}_0$ in a resummed diagram and the bond linking $\mathcal{P}_0$ to $\mathcal{P}_j$ may decay algebraically. As in Eqs. (C3) and (C4) of Ref. [1], once the invariance of both the integration measure and the integrand under the inversion $\mathbf{X}_j \to -\mathbf{X}_j$ has been taken into account, the only terms at the border of integrability come from the case $\mathcal{P}_0 = \mathcal{L}_a$ (where $\mathcal{L}_a$ is the root point). These terms are proportional to

$$\int D(\mathbf{X}_j) G_2(\mathbf{X}_j) [\mathbf{X}_j(\tau')]_\mu^2 \partial_{\mu\mu} \left(\frac{1}{r} \right), \tag{38}$$

where the function $G_2(\mathbf{X}_j)$ is invariant under rotations around the direction of $\mathbf{B}_0$. In Eq. (38) the summation over the repeated space index $\mu = 1,2,3$ is implicit and $\mathbf{r} = \mathbf{\Omega}_a(\tau) - \mathbf{R}_j$. Because of the invariance of $D(\mathbf{X}_j) G_2(\mathbf{X}_j)$ under rotations in the plane perpendicular to $\mathbf{B}_0$, Eq. (38) can be written as

$$\{ A(\tau')\Delta + [B(\tau') - A(\tau')] \partial_{zz} \} \left(\frac{1}{r} \right), \tag{39}$$

with $A(\tau') = \int D(\mathbf{X}_j) G_2(\mathbf{X}_j) [\mathbf{X}_j(\tau')]_x^2$ and $B(\tau') = \int D(\mathbf{X}_j) G_2(\mathbf{X}_j) [\mathbf{X}_j(\tau')]_z^2$. However, after integration over the unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$,

$$\int d\hat{\mathbf{r}} \, \partial_{zz} \left(\frac{1}{r} \right) = \frac{1}{3} \Delta \left(\frac{1}{r} \right) = -\frac{4\pi}{3} \delta(\mathbf{r}), \tag{40}$$

where $\delta(\mathbf{r})$ is the Dirac distribution. Finally, Eq. (38) is short ranged after integration over $\hat{\mathbf{r}}$.

Since the resummed diagrams in the loop-fugacity expansion of $\rho(\mathcal{L})$ are finite, we can use the diagrammatic relation (27) in which the weight of each loop is equal to the loop density. This relation has not been studied in Ref. [1]. Again, the Coulomb divergencies of the diagrams $\mathbb{G}^*$ can be resummed by a procedure analogous to that performed for the Ursell function. The reason is that the diagrams $\mathbb{G}^*$ do not contain any articulation point, except for the diagram with only one internal point (in which the root point $\mathcal{L}_a$ is itself an

articulation point). The special role of the single root point introduces two differences with the case of the Ursell function, as follows. Let us define f^{cc} (f^{mc}) as F^{cc} (F^{mc}) with v_C in place of ϕ in Eqs. (30) and (32). First, there appears an extra constant

$$\frac{1}{2}\int d\mathcal{L}\rho(\mathcal{L})\{[f^{cc}+f^{mc}][F^{cc}+F^{mc}]-[f^{cc}]^2\}(\mathcal{L}_a,\mathcal{L}) \tag{41}$$

due to the resummation of ring diagrams with Coulomb intermediate points, because these diagrams disappear in the resummation process, as ring diagrams in the loop-fugacity expansion of $\rho(\mathcal{L}_a)$ [see Eq. (B8) in Ref. [1]]. The term $[f^{cc}]^2(\mathcal{L}_a,\mathcal{L})$ must be subtracted because there is no ring with only one internal point. (Indeed, two points are linked by at most one bond f.) Second, in the diagram with only one internal point $\mathcal{L}$ and a bond $F_R(\mathcal{L}_a,\mathcal{L})$, one must subtract the contributions that are Coulomb rings in order to avoid double counting, as in Eq. (B3) of Ref. [1]. The redundant contribution that must be subtracted is equal to

$$\frac{1}{2}\int d\mathcal{L}\rho(\mathcal{L})\{[F^{cc}+F^{mc}]^2-[f^{cc}]^2\}(\mathcal{L}_a,\mathcal{L}). \tag{42}$$

The spurious infinite contribution of $[f^{cc}]^2$ disappears in the difference between Eqs. (41) and (42). If we denote α the species of the root point $\mathcal{L}_a$ with exchange degeneracy p_a and γ the species of the internal point $\mathcal{L}$, the difference between Eqs. (41) and (42), which will be denoted $I_{\mathrm{rg}T}$ (since it comes from some truncated contribution of Coulomb rings), may be written as

$$I_{\mathrm{rg}T}=\frac{\kappa^2}{8\pi}\beta(e_\alpha p_a)^2\int d\mathbf{r}\int_0^{p_a}\frac{d\tau}{p_a}\sum_{q=0}^{\infty}\frac{1}{q!}[\mathbf{X}_a(\tau)\cdot\boldsymbol{\nabla}_{\mathbf{r}}]^q \times\left(\frac{1-e^{-\kappa r}}{r}\right)\int_0^{p_a}\frac{d\tau'}{p_a}\sum_{q'=0}^{\infty}\frac{1}{q'!}[\mathbf{X}_a(\tau')\cdot\boldsymbol{\nabla}_{\mathbf{r}}]^{q'}\left(\frac{e^{-\kappa r}}{r}\right), \tag{43}$$

where the factor κ^2 given by Eq. (34) originates from the integration over the internal degrees of freedom of $\mathcal{L}$. The resummed diagrams with at least two internal points are obtained by the same resummation process as in the case of the Ursell function. [We notice that the resummation for the loop-fugacity expansion of $\rho(\mathcal{L}_a)$ performed in Appendix B of Ref. [1] is more complex because the existence of articulation points in the loop-fugacity diagrams leads to the introduction of two types of weights after resummations as well as related excluded-convolution rules in order to avoid double counting.] The final formula for $\rho(\mathcal{L}_a)$ after resummation reads

$$\rho(\mathcal{L}_a)=z(\mathcal{L}_a)\exp[I_{\mathrm{rg}T}(\mathcal{L}_a)] \times\exp\left[\sum_{\mathbb{P}^*}\frac{1}{S_{\mathbb{P}^*}}\int\prod_{m=1}^{M}d\mathcal{P}_m\rho(\mathcal{P}_m)\left[\prod F\right]_{\mathbb{P}^*}\right], \tag{44}$$

where the diagrams $\mathbb{P}^*$ contain the root point $\mathcal{L}_a$ and at least one internal point. They are built with the bonds F^{cc}, F^{cm}, F^{mc}, and F_R and obey the same topological properties as the diagrams $\mathbb{G}^*$ in Eq. (27) with the extra excluded-convolution rule also valid for the Ursell function.

V. A SOLVABLE MODEL

We consider two quantum charges e_1 and e_2 at points $\mathbf{r}_1$ and $\mathbf{r}_2$ embedded in a classical plasma. This model was first introduced by Alastuey and Martin [19] in order to exhibit how quantum fluctuations generate algebraic tails in position correlations at large distances. In this section we study the generalization of this model in the presence of a uniform magnetic field $\mathbf{B}_0$.

A. Definition of the model

In order to define the correlation between the two quantum particles from the free energies associated with their immersion in the classical gas either separately or together, we decompose the Hamiltonian of the whole system as $H=H_0(C)+H(1,C)+H(2,C)+e_1e_2v_C(\mathbf{r}_1-\mathbf{r}_2)$ with the following definitions. $H_0(C)$ is the Hamiltonian of the classical plasma in a phase-space configuration $C=(\{\mathbf{y}_j\}_{j=1,\ldots,N},\{\mathbf{p}_j\}_{j=1,\ldots,N})$ of its N particles in the absence of the quantum charges,

$$H_0(C)=\sum_{j=1}^{N}\frac{[\mathbf{p}_j-(e_j/2c)\mathbf{B}_0\wedge\mathbf{y}_j]^2}{2m_j}+\frac{1}{2}\int d\mathbf{r}\int d\mathbf{r}' \times v_C(\mathbf{r}-\mathbf{r}')Q(\mathbf{r},C)Q(\mathbf{r}',C), \tag{45}$$

where $Q(\mathbf{r},C)$ is the microscopic charge density at $\mathbf{r}$ of the classical gas in the configuration C. $H(i,C)$ is the Hamiltonian of one quantum charge with index $i=1,2$ in the potential created by the classical plasma in configuration C,

$$H(i,C)=\frac{[\mathbf{p}_i-(e_i/2c)\mathbf{B}_0\wedge\mathbf{r}_i]^2}{2m_i}+e_i\int d\mathbf{r}\,v_C(\mathbf{r}_i-\mathbf{r})Q(\mathbf{r},C). \tag{46}$$

In a rigorous approach, one must first consider a system in a finite volume Λ and then take the thermodynamic limit. In the following we consider a system in an infinite volume from the beginning, because this does not change the results.

As in Ref. [19], the correlation is defined from the immersion free energies as

$$g(\mathbf{r}_1,\mathbf{r}_2)\equiv\exp\{-\beta[F_{12}^{(2)}(\mathbf{r}_1,\mathbf{r}_2)-F_1^{(1)}(\mathbf{r}_1)-F_2^{(1)}(\mathbf{r}_2)]\}-1, \tag{47}$$

where $F_i^{(1)}(\mathbf{r}_i)$ is the free energy associated with the immersion of one quantum charge with species i at point $\mathbf{r}_i$ in the classical gas

$$e^{-\beta F_i^{(1)}(\mathbf{r}_i)}\equiv\frac{\int dC\langle\mathbf{r}_i|e^{-\beta[H_0(C)+H(i,C)]}|\mathbf{r}_i\rangle}{\int dC e^{-\beta H_0(C)}} \tag{48}$$

and $F_{12}^{(2)}(\mathbf{r}_1,\mathbf{r}_2)$ is the free energy associated with the immersion of the pair of quantum particles at positions $\mathbf{r}_1$ and $\mathbf{r}_2$,

$$e^{-\beta F_{12}^{(2)}(\mathbf{r}_1,\mathbf{r}_2)} \equiv \frac{\int dC \langle \mathbf{r}_1,\mathbf{r}_2| e^{-\beta H(C)} |\mathbf{r}_1,\mathbf{r}_2\rangle}{\int dC e^{-\beta H_0(C)}}. \quad (49)$$

In Eqs. (48) and (49) the configurations of classical particles are integrated over with $dC \equiv \Pi_{j=1}^{N} d\mathbf{y}_j d\mathbf{p}_j/(2\pi\hbar)^{3N}$. We notice that in fact $H_0(C)$ is a scalar which factorized out of the matrix elements. In fact, since the position integrals are performed over an infinite volume, $F_i^{(1)}(\mathbf{r}_i)$ is independent from $\mathbf{r}_i$, $F_i^{(1)}(\mathbf{r}_i) = F_i^{(1)}$, while $F_{12}^{(2)}(\mathbf{r}_1,\mathbf{r}_2)$ only depends on the difference $\mathbf{r}_2 - \mathbf{r}_1$.

B. Averaging over the classical gas

In the averaging process, the mechanism underlying the Bohr–van Leeuwen theorem still operates and there is no macroscopic magnetic property associated with the classicalparticles. Indeed, the conjugate momentum $\mathbf{p}_j$ of a classical particle is a scalar and, by a translation $\mathbf{p}_j \to \mathbf{p}_j + (e_j/2c)\mathbf{B}_0 \wedge \mathbf{r}_j$, the coupling with $\mathbf{B}_0$ disappears for the degrees of freedom of classical particles, as follows:

$$\int dC e^{-\beta H_0(C)} = \left[\prod_{j=1}^{N} \frac{1}{(2\pi\lambda_j^2)^{3/2}}\right] \int \left[\prod_j d\mathbf{r}_j\right] \times \exp\left[-(\beta/2)\sum_{j\neq l} e_j e_l v_C(\mathbf{r}_j - \mathbf{r}_l)\right]. \quad (50)$$

On the contrary, the position and momentum operators do not commute for quantum particles. This property is reflected in the path integral representation in the phase space $(\mathbf{r},\mathbf{p})$ [20] by the fact that, when the paths of the variable $\mathbf{p}$ are integrated over first, there appears a phase factor which couples the paths of the variable $\mathbf{r}$ with $\mathbf{B}_0$. For a closed path $\boldsymbol{\omega}_{i,i} = \mathbf{r}_i + \lambda_i \boldsymbol{\xi}(s)$, the phase factor involves $\int_{\text{Itô}} \boldsymbol{\omega}_{i,i} \wedge d\boldsymbol{\omega}_{i,i} = \lambda_i^2 \int_{\text{Itô}} \boldsymbol{\xi}_i \wedge d\boldsymbol{\xi}_i$, because $\int_{\text{Itô}} d\boldsymbol{\xi} = \mathbf{0}$ according to Itô lemma (16). Then the generalization of Eq. (17) in the presence of an external potential allows one to write

$$\langle \mathbf{r}_i| e^{-\beta H(i,C)} |\mathbf{r}_i\rangle = \frac{1}{(2\pi\lambda_i^2)^{3/2}} \int D(\boldsymbol{\xi}_i) \times \exp\left[i(e_i\lambda_i^2/2\hbar c)\mathbf{B}_0 \cdot \int_{\text{Itô}} \boldsymbol{\xi}_i \wedge d\boldsymbol{\xi}_i \right] \times e^{-\beta e_i \sum_{j=1}^{N} \int_0^1 ds v_C(\mathbf{r}_i + \lambda_i \boldsymbol{\xi}_i(s) - \mathbf{y}_j)}. \quad (51)$$

As in the general formalism, the closed path $\boldsymbol{\omega}_{i,i}$ may be interpreted as a closed curve with a uniform charge density $e_i n_i(\mathbf{r}) = e_i \int_0^1 ds\, \delta(\mathbf{r}_i + \lambda_i \boldsymbol{\xi}_i(s) - \mathbf{r})$.

In the case of the one-body quantum Gibbs factor that appears in Eq. (48), the use of the Feynman-Kac-Itô formula (51) introduces the electrostatic interaction of the closed curve $\boldsymbol{\omega}_{i,i}$ with a given configuration of the classical particles. Henceforth, after averaging over the classical gas configurations, there appears the electrostatic free energy $F_{i,\text{elect}}^{(1)}(\boldsymbol{\xi}_i)$ of the immersion of a single closed curve in the classical gas,

$$e^{-\beta F_i^{(1)}} = \frac{1}{(2\pi\lambda_i^2)^{3/2}} \int D(\boldsymbol{\xi}_i) \times \exp\left[(ie_i\lambda_i^2/2\hbar c)\mathbf{B}_0 \cdot \int_0^1 \boldsymbol{\xi}_i(s) \wedge d\boldsymbol{\xi}_i \right] e^{-\beta F_{i,\text{elect}}^{(1)}(\boldsymbol{\xi}_i)}, \quad (52)$$

with

$$e^{-\beta F_{i,\text{elect}}^{(1)}(\boldsymbol{\xi}_i)} = \frac{\int \left[\prod_j d\mathbf{y}_j\right] \exp\left[-\beta e_i \sum_j e_j \int d\mathbf{r} n_i(\mathbf{r}) v_C(\mathbf{r} - \mathbf{y}_j) - (\beta/2)\sum_{j\neq l} e_j e_l v_C(\mathbf{y}_j - \mathbf{y}_l)\right]}{\int \left[\prod_j d\mathbf{y}_j\right] \exp\left[-(\beta/2)\sum_{j\neq l} e_j e_l v_C(\mathbf{y}_j - \mathbf{y}_l)\right]}. \quad (53)$$

$F_{i,\text{elect}}^{(1)}(\boldsymbol{\xi}_i)$ is independent from the position $\mathbf{r}_i$ of the closed curve because the classical gas occupies an infinite volume.

On the contrary, in the case of the two-body quantum Gibbs factor involved in Eq. (49), the pair interaction that appears in the Feynman-Kac-Itô formula (17), namely, $e_1 e_2 \int_0^1 ds_1 \int_0^1 ds_2 v_C(\mathbf{r}_2 - \mathbf{r}_1 + \lambda_2 \boldsymbol{\xi}_2(s_2) - \lambda_1 \boldsymbol{\xi}_1(s_1))$, is not an electrostatic energy. However, it can be written as the sum of the purely electrostatic contribution $E_{\text{elect}}(\mathbf{r}_1 - \mathbf{r}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$, which couples every curve element of one closed curve with all curve elements of the other closed curve, and a purely quantum term,

$$w(\mathbf{r}_1 - \mathbf{r}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \equiv e_1 e_2 \int_0^1 ds_1 \int_0^1 ds_2 [\delta(s_1 - s_2) - 1] v_C(\mathbf{r}_2 - \mathbf{r}_1 + \lambda_2 \boldsymbol{\xi}_2(s_2) - \lambda_1 \boldsymbol{\xi}_1(s_1)) \quad (54)$$

(w corresponds to the quantity denoted by W in Ref. [19]). With this decomposition, the pair free energy reads

$$e^{-\beta F_{12}^{(2)}(\mathbf{r}_1 - \mathbf{r}_2)} = \frac{1}{(2\pi\lambda_1^2)^{3/2}(2\pi\lambda_2^2)^{3/2}} \int D(\boldsymbol{\xi}_1) \exp\left[(ie_1\lambda_1^2/2\hbar c)\mathbf{B}_0 \cdot \int_0^1 \boldsymbol{\xi}_1(s) \wedge d\boldsymbol{\xi}_1 \right] \times \int D(\boldsymbol{\xi}_2) \exp\left[(ie_2\lambda_2^2/2\hbar c)\boldsymbol{B}_0 \cdot \int_0^1 \boldsymbol{\xi}_2(s) \wedge d\boldsymbol{\xi}_2 \right] e^{-\beta w(\mathbf{r}_1 - \mathbf{r}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)} e^{-\beta F_{12,\text{elect}}^{(2)}(\mathbf{r}_1 - \mathbf{r}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2)}, \quad (55)$$

where $F^{(2)}_{12,\mathrm{elect}}(\mathbf{r}_1-\mathbf{r}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ is the free energy associated with the immersion of two closed curves interacting through the electrostatic force. As in Ref. [19], we introduce the effective potential corresponding to the electrostatic energy needed to separate the charged filaments at $\mathbf{r}_1$ and $\mathbf{r}_2$ by an infinite distance in the classical gas,

$$\phi_{\mathrm{eff}}(\mathbf{r}_1-\mathbf{r}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)=F^{(2)}_{12,\mathrm{elect}}(\mathbf{r}_1-\mathbf{r}_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)-F^{(1)}_{1,\mathrm{elect}}(\boldsymbol{\xi}_1)-F^{(1)}_{2,\mathrm{elect}}(\boldsymbol{\xi}_2). \tag{56}$$

Subsequently, with $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, the correlation (47) can be written as

$$g(\mathbf{r})=\int \overline{D_{\mathbf{B}_0}(\boldsymbol{\xi}_1)}\int \overline{D_{\mathbf{B}_0}(\boldsymbol{\xi}_2)}\{e^{-\beta[\phi_{\mathrm{eff}}+w]}-1\}, \tag{57}$$

with

$$\overline{D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)}\equiv\frac{\exp\left[(ie_i\lambda_i^2/2\hbar c)\mathbf{B}_0\cdot\int_0^1\boldsymbol{\xi}_i(s)\wedge d\boldsymbol{\xi}_i\right]e^{-\beta F^{(1)}_{i,\mathrm{elect}}(\boldsymbol{\xi}_i)}}{\int D(\boldsymbol{\xi}_i)\exp\left[(ie_i\lambda_i^2/2\hbar c)\mathbf{B}_0\cdot\int_0^1\boldsymbol{\xi}_i(s)\wedge d\boldsymbol{\xi}_i\right]e^{-\beta F^{(1)}_{i,\mathrm{elect}}(\boldsymbol{\xi}_i)}}D(\boldsymbol{\xi}_i). \tag{58}$$

The large-distance behavior of $g(\mathbf{r})$ can be easily investigated from Eq. (57). Every Brownian bridge $\lambda_i\boldsymbol{\xi}_i$ has a Gaussian weight that restricts its average extent to distances of order λ_i. Besides, by virtue of the exponential screening in classical Coulomb systems [21], ϕ_{eff} decays faster than any inverse power law of the distance r, whereas w falls off algebraically. More precisely, according to the Taylor formula

$$v_C[\mathbf{r}_2-\mathbf{r}_1+\lambda_2\boldsymbol{\xi}_2(s_2)-\lambda_1\boldsymbol{\xi}_1(s_1)]=\sum_{n=0}^{\infty}(1/n!)\{[\lambda_2\boldsymbol{\xi}_2(s_2)-\lambda_1\boldsymbol{\xi}_1(s_1)]\cdot\boldsymbol{\nabla}\}^n(1/r),$$

and according to the property

$$\int_0^1 ds_1\int_0^1 ds_2[\,\delta(s_1-s_2)-1]f(s_1)=0, \tag{59}$$

the leading algebraic tail of w decays as $1/r^3$, and w^2 falls off as $1/r^6$.

Subsequently, algebraic tails appear in the large-distance behavior of $g(\mathbf{r})$ and are given by

$$\int\overline{D_{\mathbf{B}_0}(\boldsymbol{\xi}_1)}\int\overline{D_{\mathbf{B}_0}(\boldsymbol{\xi}_2)}\left\{-\beta w(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)+\frac{\beta^2}{2}[w(\mathbf{r},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)]^2+\cdots\right\}. \tag{60}$$

After averaging $\{\cdots\}$ in Eq. (60) with the measure $\overline{D_{\mathbf{B}_0}(\boldsymbol{\xi})}$, which is invariant under the inversion $\boldsymbol{\xi}_i\to-\boldsymbol{\xi}_i$, the slowest nonvanishing term in Eq. (60) is the term in the Taylor decomposition of w that contains two $\boldsymbol{\xi}_1$'s and two $\boldsymbol{\xi}_2$'s. This term decays as $1/r^5$, whereas, in the absence of $\mathbf{B}_0$, the invariance under rotations makes this term proportional to $\Delta(1/r)$, which is short-ranged (see Ref. [5]). We define the covariance in the presence of the classical gas as

$$\overline{\mathrm{cov}^{\alpha_i}_{\mu\nu}}(s,s')\equiv\int\overline{D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)}[\boldsymbol{\xi}_i(s)]_\mu[\boldsymbol{\xi}_i(s')]_\nu. \tag{61}$$

This covariance depends on the species α_i of the particle i through the phase factor and the electrostatic free energy in Eq. (58). With this notation, the leading algebraic tail in Eq. (60) is

$$g(\mathbf{r})\underset{r\to\infty}{\sim}-\frac{1}{4}\beta e_1e_2\lambda_1^2\lambda_2^2\int_0^1 ds_1\int_0^1 ds_2\,[\,\delta(s_1-s_2)-1]\times\overline{\mathrm{cov}}^{\alpha_1}_{\mu\nu}(s_1,s_1)\overline{\mathrm{cov}}^{\alpha_2}_{\rho\sigma}(s_2,s_2)\partial_{\mu\nu\rho\sigma}\left(\frac{1}{r}\right). \tag{62}$$

C. Covariance properties

In order to give a more explicit expression for the asymptotic behavior of $g(\mathbf{r})$, we briefly present the properties of the covariance that are deduced from those of the measure $\overline{D_{\mathbf{B}_0}(\boldsymbol{\xi}_i)}$ defined in Eq. (58). The term $\mathbf{B}_0\cdot\int\boldsymbol{\xi}\wedge d\boldsymbol{\xi}$ in the phase factor involves only the components of $\boldsymbol{\xi}$ that lie in the plane perpendicular to $\mathbf{B}_0$.

First, the phase factor is invariant under rotations of $\boldsymbol{\xi}$ in this plane. Therefore

$$\overline{\mathrm{cov}}^{\alpha_i}_{xx}(s,s')=\overline{\mathrm{cov}}^{\alpha_i}_{yy}(s,s'), \tag{63}$$

$$\overline{\mathrm{cov}}^{\alpha_i}_{xy}(s,s')=-\overline{\mathrm{cov}}^{\alpha_i}_{yx}(s,s'). \tag{64}$$

Since by definition $\overline{\mathrm{cov}}^{\alpha_i}_{xy}(s,s)=\overline{\mathrm{cov}}^{\alpha_i}_{yx}(s,s)$, Eq. (64) implies that

$$\overline{\mathrm{cov}}^{\alpha_i}_{xy}(s,s)=0. \tag{65}$$

Second, the phase factor is unchanged when $[\boldsymbol{\xi}]_z\to-[\boldsymbol{\xi}]_z$, and

$$\overline{\mathrm{cov}}^{\alpha_i}_{zx}(s,s')=\overline{\mathrm{cov}}^{\alpha_i}_{xz}(s,s')=0, \tag{66}$$

$$\overline{\mathrm{cov}}^{\alpha_i}_{zy}(s,s')=\overline{\mathrm{cov}}^{\alpha_i}_{yz}(s,s')=0. \tag{67}$$

As a result of Eq. (65) and of the latter equations

$$\overline{\mathrm{cov}}^{\alpha_i}_{\mu\nu}(s,s)=\delta_{\mu,\nu}\overline{\mathrm{cov}}^{\alpha_i}_{\mu\mu}(s,s). \tag{68}$$

In order to rewrite Eq. (62) with two operators analogous to $\{\cdots\}$ in Eq. (39), we use Eq. (68) and (63). Since

$$\frac{\partial^n}{(\partial z)^n}\left(\frac{1}{r}\right)=(-1)^n n!\frac{P_n(\cos\theta)}{r^{n+1}}, \tag{69}$$

where P_n is the Legendre polynomial of order n and θ is the angle between $\mathbf{r}$ and $\mathbf{B}_0$, we find that $g(r)$ decays as

$$g(r)\underset{r\to\infty}{\sim}-6\frac{P_4(\cos\theta)}{r^5}\beta e_1 e_2 \lambda_1^2\lambda_2^2\int_0^1 ds_1\int_0^1 ds_2 \times[\delta(s_1-s_2)-1][\overline{\mathrm{cov}}_{xx}^{\alpha_1}(s_1,s_1)-\overline{\mathrm{cov}}_{zz}^{\alpha_1}(s_1,s_1)] \times[\overline{\mathrm{cov}}_{xx}^{\alpha_2}(s_2,s_2)-\overline{\mathrm{cov}}_{zz}^{\alpha_2}(s_2,s_2)]. \tag{70}$$

In conclusion, this model is solvable, and the exact algebraic tail of the correlation is given by Eq. (70). Since $\int d(\cos\theta)P_n(\cos\theta)=0$, after integration over θ, $g(r)$ decays as $1/r^6$, according to Eq. (60), as in the absence of $\mathbf{B}_0$. In the limit of weak Coulomb coupling and weak dynamical effects (at $u_{Ci}=\beta\hbar e_i B_0/2m_i c$ fixed), the coefficient of the $1/r^5$ tail can be calculated exactly. The result will be given in Paper III.

VI. LEADING ALGEBRAIC TAILS OF THE STATIC CORRELATIONS AT ANY DENSITY

The scheme of the discussion is the same as in Ref. [5], but now the arguments about invariance under rotations must be decomposed in arguments about either invariance under inversion or invariance under rotations in the plane perpendicular to $\mathbf{B}_0$. The results derived in Ref. [5] and that depend only on the invariance under inversion are still valid. However, the invariance under rotations is broken by $\mathbf{B}_0$ and the discussion about analytical properties in Fourier space is modified.

A. Scheme of the discussion

We consider two charges with species α_a and α_b. The two-body distribution function (called correlation in the following) can be decomposed into two contributions according to the general formalism of Ref. [5]. If $\alpha_a=\alpha_b$, the so-called exchange part $\rho^{(2)T}_{\alpha_a\alpha_a}|_{\mathrm{exch}}$ comes from configurations where the positions of the particles are involved in the same cyclic permutation. It is determined by integration of the loop density over all its internal degrees of freedom except for the relative distance between two particles in the loop [see Eq. (4.6) of Ref. [1]]. The exchange contribution decays faster than any inverse power law of the distance, because the phase factor originating from the magnetic field does not modify the argument in Sec. V D of Ref. [1]. The other part $\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathrm{nonexch}}$ of the correlation is calculated by integration of the loop correlation over the internal degrees of freedom of the loops (see Eq. (4.7) of Ref. [1]). In the thermodynamic limit the loop density does not depend on the position of the loop and $\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathrm{nonexch}}$ can be written in terms of the Ursell function as

$$\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathrm{nonexch}}(\mathbf{r})=\sum_{p_a}\sum_{p_b}p_a p_b\int D(\mathbf{X}_a)\rho(\chi_a) \times\int D(\mathbf{X}_b)\rho(\chi_b)h(\mathbf{r},\chi_a,\chi_b), \tag{71}$$

where χ is a global notation for the internal degrees of freedom of a loop, $\chi=(\alpha,p,\mathbf{X})$, and $\rho(\chi)\equiv\rho_{\alpha,p}(\mathbf{X})$. The large-distance behavior of $\rho^{(2)T}_{\alpha_a\alpha_b}|_{\mathrm{nonexch}}$ is analyzed by a reorganization of diagrams in order to exhibit the properties arising from the structure of W.

As in Sec. I B of Ref. [5], we introduce other diagrams called $\tilde{\Pi}$, by splitting the resummed bond F_R into two bonds W and F_{R6},

$$F_R=W+F_{R6}. \tag{72}$$

The point of the decomposition (72) is that the leading asymptotic behavior of F_{R6} falls off as $1/r^6$ at large distances. The representation of $h(\mathbf{r},\chi_a,\chi_b)$ in terms of diagrams $\tilde{\Pi}$ is the same as that given in Eq. (29) and diagrams $\tilde{\Pi}$ have the same properties as diagrams Π. Let H denote the sum of the so-called $\tilde{\Pi}_{W_c}$ diagrams that remain connected when a bond W is cut. According to some kind of Dyson equation (which also appears in the definition of the ''direct'' correlation function) $h(\mathbf{r},\chi_a,\chi_b)$ is equal to a series of convolutions involving H and W. If we denote $g(\mathbf{k})=\int d\mathbf{r}\exp[i\mathbf{k}\cdot\mathbf{r}]g(\mathbf{r})$, the series reads in Fourier space

$$\rho(\chi_a)h(\mathbf{k},\chi_a,\chi_b)\rho(\chi_b) =H(\mathbf{k},\chi_a,\chi_b)+\sum_{I=1}^{\infty}\int d\chi_1\cdots d\chi_I\, d\chi_1'\cdots d\chi_I' \times K(\mathbf{k},\chi_a,\chi_1)\times W(\mathbf{k},\chi_1,\chi_1')K(\mathbf{k},\chi_1',\chi_2)\cdots \times W(\mathbf{k},\chi_I,\chi_I')K(\mathbf{k},\chi_I',\chi_b), \tag{73}$$

where

$$K(\mathbf{R}_i-\mathbf{R}_j,\chi_i,\chi_j)\equiv\delta(\mathbf{R}_i-\mathbf{R}_j)\delta_{\chi_i,\chi_j}\rho(\chi_i) +H(\mathbf{R}_i-\mathbf{R}_j,\chi_i,\chi_j),$$

with $\delta_{\chi_i,\chi_j}\equiv\delta_{\alpha_i,\alpha_j}\delta_{p_i,p_j}\delta(\mathbf{X}_i-\mathbf{X}_j)$. The representation of h in terms of the graph H and a sum of chains made with graphs K linked by I bonds W can be written with short notations as

$$\rho h\rho=H+K*W*K+K*W*K*W*K+\cdots. \tag{74}$$

This decompositon is useful because H, and subsequently K, decays as $1/r^6$ for topological reasons (see Sec. III A of Ref. [5]), even before integration over the shapes of the root points $\mathcal{L}_a$ and $\mathcal{L}_b$. Moreover, the dimensional analysis and the invariance under inversion show that the convolutions in Eq. (73) fall off at least as $1/r^5$. In fact, when $\mathbf{B}_0=\mathbf{0}$, because of the invariance under rotation, there appear powers of the Laplacian and contributions that would decay as $1/r^5$ according to the sole dimensional analysis are in fact short ranged because of the harmonicity of the Coulomb potential.

The general mechanism was discussed in Sec. I B of Ref. [5] and it is exemplified in the solvable model of Sec. V where the leading asymptotic behavior is in fact given by $\beta^2 w^2 = W^2$ according to Eq. (60). When $\mathbf{B}_0 \neq \mathbf{0}$, there is no invariance under rotation and convolutions indeed decay as $1/r^5$, as in the solvable model where the correlation is given by $-\beta w = W$.

The following discussion is organized in two steps. First, we give the slowest possible decay of the convolutions in Eq. (73) that is derived from dimensional analysis and invariance under inversion. Since the term ''rotational invariance'' used in Sec. III B of Ref. [5] is too restrictive and can be replaced by ''invariance under inversion,'' the result about the minimal inverse power law is the same in the presence or in the absence of the magnetic field. However, the exponent of the slowest decay is altered by the existence of $\mathbf{B}_0$, because it depends crucially on rotational invariance arguments. This point will be discussed in the second step.

B. Dimensional analysis and invariance under inversion

The large-distance behavior of the convolutions in Eq. (73) is derived from a Fourier transform analysis according to the principles presented in Sec. II C of Ref. [5]. In short, the algebraic tails of a function g at large distances are exactly given by the inverse Fourier transforms of the terms in the small-$\mathbf{k}$ expansion of $g(\mathbf{k})$ that are nonanalytic in the components of $\mathbf{k}$. Subsequently, the leading large-distance decay of a convolution $g_1 * g_2$ is easily determined: it is merely given by the nonanalytic terms in the small-$\mathbf{k}$ expansion of the product of the small-$\mathbf{k}$ expansions of g_1 and g_2 that are of the lowest order in $|\mathbf{k}|$. For instance, the asymptotic behavior of the inverse Fourier transform of the product $g_1(\mathbf{k})g_2(\mathbf{k})$ is given by that of the two singular terms $g_1(\mathbf{k}=\mathbf{0})S_{g_2}(\mathbf{k})$ and $g_2(\mathbf{k}=\mathbf{0})S_{g_1}(\mathbf{k})$, which is of the lowest order and does not vanish. If $S_{g_1}(\mathbf{k})$ is of order zero in $|\mathbf{k}|$, an extra singular term $S_{g_1}(\mathbf{k})S_{g_2}(\mathbf{k})$ appears. If S_{g_1} and S_{g_2} are of the same order in $|\mathbf{k}|$, then the asymptotic behavior is given by the sum of the terms involving $S_{g_1}(\mathbf{k})$ or/and $S_{g_2}(\mathbf{k})$.

In order to distinguish the various (leading and subleading) algebraic tails, we introduce the following decomposition of W derived from Eq. (36):

$$W(\mathbf{k},\chi_i,\chi_i') = -\beta e_{\alpha_i} e_{\alpha_i'} \int_0^{p_i} d\tau_i \int_0^{p_i'} d\tau_i' \{\delta([\tau_i - P(\tau_i)] - [\tau_i' - P(\tau_i')]) - 1\} \sum_{m_i=1}^{\infty} \sum_{n_i=1}^{\infty} \frac{1}{m_i! n_i!} \times w^{[m_i,n_i]}[\mathbf{k},\mathbf{X}_i(\tau_i),\mathbf{X}_i'(\tau_i')], \tag{75}$$

where $w^{[m_i,n_i]}$ is a singular term of order $|\mathbf{k}|^{m_i+n_i-2}$,

$$w^{[m_i,n_i]}[\mathbf{k},\mathbf{X}_i(\tau_i),\mathbf{X}_i'(\tau_i')] \equiv [i\mathbf{X}_i(\tau_i)\cdot\mathbf{k}]^{m_i} \times [-i\mathbf{X}_i'(\tau_i')\cdot\mathbf{k}]^{n_i} \frac{4\pi}{\mathbf{k}^2}. \tag{76}$$

The relation (73) may be written as a series of chains, each of which involves I terms $w^{[m_i,n_i]}$. After integration over the shapes of the root points $\mathcal{L}_a$ and $\mathcal{L}_b$, these chains are denoted by $\mathcal{C}_I(\mathbf{k};\{m_i\},\{n_i\})$ as in Sec. III B of Ref. [5].

In order to simply discuss symmetry arguments, we also introduce $\mathcal{K}_{i,i+1}^{[n_i,m_{i+1}]}$:

$$\mathcal{K}_{i,i+1}^{[n_i,m_{i+1}]}(\mathbf{k}) \equiv \int D(\mathbf{X}_i') \int D(\mathbf{X}_{i+1}) [\mathbf{k}\cdot\mathbf{X}_i'(\tau_i')]^{n_i} \times [\mathbf{k}\cdot\mathbf{X}_{i+1}(\tau_{i+1})]^{m_{i+1}} K(\mathbf{k},\chi_i',\chi_{i+1}) \tag{77}$$

and

$$\mathcal{K}_{a,1}^{[m_1]}(\mathbf{k}) \equiv \int D(\mathbf{X}_a) \int D(\mathbf{X}_1) [\mathbf{k}\cdot\mathbf{X}_1(\tau_1)]^{m_1} K(\mathbf{k},\chi_a,\chi_1). \tag{78}$$

with a similar expression for $\mathcal{K}_{I,b}^{[n_I]}(\mathbf{k})$. According to these definitions, up to multiplicative factors,

$$\mathcal{C}_I(\mathbf{k};\{m_i\},\{n_i\}) \propto \frac{1}{|\mathbf{k}^2|^I} \mathcal{K}_{a,1}^{[m_1]}(\mathbf{k}) \mathcal{K}_{1,2}^{[n_1,m_2]}(\mathbf{k}) \times \mathcal{K}_{2,3}^{[n_2,m_3]}(\mathbf{k}) \cdots \mathcal{K}_{I-1,I}^{[n_{I-1}m_I]}(\mathbf{k}) \mathcal{K}_{I,b}^{[n_I]}(\mathbf{k}). \tag{79}$$

Since H decays as $1/r^6$, even before integration over the loop shapes, the first nonanalytic term in the small-$\mathbf{k}$ expansion of $H(\mathbf{k},\chi_i',\chi_{i+1})$ is of order $|\mathbf{k}|^3$ and will be called $S_H^{(3)}(\mathbf{k})$. As a result,

$$H(\mathbf{k},\chi_i',\chi_{i+1}) \underset{|\mathbf{k}|\to 0}{\sim} \int d\mathbf{r}\, H(\mathbf{r},\chi_i',\chi_{i+1}) + i\int d\mathbf{r}(\mathbf{k}\cdot\mathbf{r}) H(\mathbf{r},\chi_i',\chi_{i+1}) - \frac{1}{2}\int d\mathbf{r}(\mathbf{k}\cdot\mathbf{r})^2 H(\mathbf{r},\chi_i',\chi_{i+1}) + S_H^{(3)}(\mathbf{k},\chi_i',\chi_{i+1}) + O_{\text{anal}}^{(3)}(\mathbf{k}) + O(|\mathbf{k}|^4), \tag{80}$$

where $O_{\text{anal}}^{(n)}(\mathbf{k})$ denotes an analytic term of order $|\mathbf{k}|^n$, whereas $O(|\mathbf{k}|^n)$ is just a term of order $|\mathbf{k}|^n$. We notice, that, as explained in Ref. [5], no $\ln|\mathbf{k}|$ term appears because of the structures of W and of the algebraic tails that it induces. The first three terms in the small-$\mathbf{k}$ expansion of $\mathcal{K}_{i,i+1}^{[n_i,m_{i+1}]}(\mathbf{k})$ are analytic and they may vanish according to arguments of invariance under inversion, even when $\mathbf{B}_0 \neq \mathbf{0}$. Indeed, the property

$$K(-\mathbf{k},-\mathbf{X}_i',-\mathbf{X}_{i+1}) = K(\mathbf{k},\mathbf{X}_i',\mathbf{X}_{i+1}) \tag{81}$$

implies that

$$\mathcal{K}_{a,1}^{[m_1]}(\mathbf{k}) = O_{\text{anal}}^{(m_1+\theta(m_1))}(\mathbf{k}) + O(|\mathbf{k}|^{m_1+\theta(m_1)+2}), \tag{82}$$

where $\theta(m)=0$ if m is even and $\theta(m)=1$ if m is odd. $O_{\rm anal}^{(m_1+\theta(m_1))}(\mathbf{k})$ comes either from the first or the second term in the small-$\mathbf{k}$ expansion of K. The structure of the small-$\mathbf{k}$ expansion of $\mathcal{K}_{I,b}^{[n_I]}(\mathbf{k})$ is similar, while

$$\mathcal{K}_{i,i+1}^{[n_i,m_{i+1}]}(\mathbf{k})=O_{\rm anal}^{(n_i+m_{i+1}+\theta(n_i+m_{i+1}))}(\mathbf{k}) + O(|\mathbf{k}|^{n_i+m_{i+1}+\theta(n_i+m_{i+1})+2}). \quad (83)$$

By inspection, it can be checked that $n_i+m_{i+1}+\theta(n_i+m_{i+1})+2\leqslant n_i+m_{i+1}+3$. Henceforth, the second term in the right-hand side of Eq. (83) may arise from the nonanalytic term $S_H^{(3)}$. Its study requires a more detailed analysis of the structure of the algebraic tails, which will be done in Sec. VII.

Finally, according to Eqs. (82) and (83), the first term in the Fourier transform of a convolution (79) with I bonds W is of order $|\mathbf{k}|^{D_{\mathcal{C}_I}}$ with

$$D_{\mathcal{C}_I}=-2I+m_1+\theta(m_1)+n_I+\theta(n_I) + \sum_{i=1}^{I-1}[n_i+m_{i+1}+\theta(n_i+m_{i+1})]. \quad (84)$$

Moreover, the next term in the small-$\mathbf{k}$ expansion of Eq. (79) is of order $|\mathbf{k}|^{D_{\mathcal{C}_I}+2}$. When the m_i's and n_i's vary, $D_{\mathcal{C}_I}$ takes only even values, and its lowest value is equal to 2. Subsequently, the dimensional analysis and the invariance under inversion ensure that a convolution $\mathcal{C}_I$ decays at least as $1/r^5$, and the first subleading tail falls off at least as $1/r^7$.

C. Full or partial rotational invariance

The preceding section dealt with the part of Sec. III B that is not changed, and now we turn to the part relative to analytical properties which is modified by the presence of the magnetic field. According to the dimensional analysis, a convolution may decay as $1/r^5$ if $D_{\mathcal{C}_I}$ takes its minimal value $D_{\mathcal{C}_I,\min}=2$, namely, if $m_1=1,2$, $n_I=1,2$, and $n_i=m_{i+1}=1$ for all $i=1,\ldots,I$. Such a convolution does fall off as $1/r^5$ if the first term in its Fourier transform $\mathcal{C}_I(\mathbf{k};\{m_i\},\{n_i\})$ is nonanalytic.

In the absence of $\mathbf{B}_0$, the system is invariant under rotations around any axis, and the first analytic term in Eq. (82) or in Eq. (83) is exactly proportional to $\mathbf{k}^2$ for the $\{m_i,n_i\}_{i=1,\ldots,I}$ that give $D_{\mathcal{C}_I,\min}$. As a consequence, in the latter cases, the first term in the convolution (79) is exactly proportional to $|\mathbf{k}|^2$ and is analytical. Subsequently, according to Sec. VI B, any convolution $\mathcal{C}_I(\mathbf{r};\{m_i\},\{n_i\})$ decays at least as $1/r^7$.

In the presence of $\mathbf{B}_0$, the system is invariant under rotations around the z axis and the first-order term in $\mathcal{K}_{i,i+1}^{[n_i,m_i]}(\mathbf{k})$ is a sum of contributions of the form $|\mathbf{k}|^{2(N_i-q_i)}[\mathbf{k}]_z^{2q_i}$ with $N_i=n_i+m_{i+1}+\theta(n_i+m_{i+1})$ and $q_i=0,\ldots,N_i$. In the convolutions $\mathcal{C}_I$ for which $D_{\mathcal{C}_I}=D_{\mathcal{C}_{I,\min}}$, every $\mathcal{K}$ starts at the order $|\mathbf{k}|^2$ by a sum of two terms which are proportional to $\mathbf{k}^2$ and $[\mathbf{k}]_z^2$ respectively. Thus, after expanding the product of the small-$\mathbf{k}$ expansions of the $\mathcal{K}$'s, the first term in Eq. (79) contains nonanalytic contributions

$$\frac{1}{(\mathbf{k}^2)^I}(\mathbf{k}^2)^{I+1-Q}([\mathbf{k}]_z^2)^Q=|\mathbf{k}|^2(\cos\theta_{\mathbf{k}})^{2Q} \quad (85)$$

as soon as $Q\geqslant 2$. ($\theta_{\mathbf{k}}$ is the angle between $\mathbf{B}_0$ and $\mathbf{k}$.) Thus a $1/r^5$ falloff, with an angular dependence, may appear.

The integration over the orientation of $\mathbf{k}$ restores the analyticity of the first term in the expansion of $\mathcal{C}_I(\mathbf{k};\{m_i\},\{n_i\})$. Moreover,

$$\int d\hat{\mathbf{r}}\, f(\mathbf{r})=\int d\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{r}}\int d\hat{\mathbf{k}}\, f(|\mathbf{k}|,\hat{\mathbf{k}}\cdot\hat{\mathbf{B}}_0), \quad (86)$$

where $\hat{\mathbf{r}}\equiv\mathbf{r}/|\mathbf{r}|$ and $\hat{\mathbf{k}}\equiv\mathbf{k}/|\mathbf{k}|$. Thus, after integration over the orientation of $\mathbf{r}$, the convolutions decay in fact as $1/r^7$.

As a conclusion, in the absence of $\mathbf{B}_0$, the particle-particle correlation decays as $1/r^6$ and this tail comes from H, whereas, in the presence of $\mathbf{B}_0$, the particle-particle correlation falls off as $1/r^5$ and this tail originates from the convolutions (while the $1/r^6$ subleading tail arises from H). After integration over angles, the $1/r^5$ tail disappears and the leading order is given by the $1/r^6$ tail coming from H.

VII. STRUCTURE OF LEADING AND SUBLEADING ALGEBRAIC TAILS OF DIAGRAMS

The leading and subleading tails of the $\tilde{\Pi}_{W_c}$ diagrams are analyzed first in order to derive the asymptotic behaviors of the convolutions $\mathcal{C}$. The result will be used extensively in Sec. VIII. As in Ref. [5], we will denote $\zeta\equiv(\alpha,p,\mathbf{Z})$ and $\zeta'\equiv(\alpha',p',\mathbf{Z}')$ the internal degrees of freedom of loops. This notation will avoid confusion of these points with the intermediate points of convolutions $\mathcal{C}$.

A. Definitions

As discussed in Sec. III D of Ref. [5], any leading or subleading algebraic tail of a diagram $\tilde{\Pi}(\mathbf{r}_a-\mathbf{r}_b,\chi_a,\chi_b)$ comes from the leading or subleading behavior of L elementary algebraic tails $S^{(\gamma_l)[q_l,q_l']}(\mathbf{r},\zeta_l,\zeta_l')$ with $l=1,\ldots,L$. By definition an elementary algebraic tail is either the asymptotic behavior of a single bond W or F_{R6} or of a convolution of diagrams $\tilde{\Pi}$ and algebraic bonds. For a convolution $\mathcal{C}$, $L=1$ and the convolution contains at least one bond W. For a $\tilde{\Pi}_{W_c}$ diagram, $L=1$ corresponds to a convolution without any bond W, whereas, when $L\geqslant 2$, the $S^{(\gamma_l)[q_l,q_l']}$'s may be convolutions involving W bonds. These elementary algebraic tails are denoted by $S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta')$ because they decay as $1/r^\gamma$ and satisfy two properties. Property (A) reads

$$\gamma=P(q,q')+q+q', \quad \text{with } \inf(q,q')\geqslant P(q,q'). \quad (87)$$

(B) corresponds to two symmetries: (1) $S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta')$ is invariant under global inversion of its arguments,

$$S^{(\gamma)[q,q']}(-\mathbf{r},-\mathbf{Z},-\mathbf{Z}')=S^{(\gamma)[q,q']}(\mathbf{r},\mathbf{Z},\mathbf{Z}') \quad (88)$$

(2) $S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta')$ is of parity $(-1)^q$ $[(-1)^{q'}]$ under the inversion $\mathbf{Z}\to-\mathbf{Z}$ $[\mathbf{Z}'\to-\mathbf{Z}']$, namely,

$$S^{(\gamma)[q,q']}(\mathbf{r},-\mathbf{Z},\mathbf{Z}')=(-1)^q S^{(\gamma)[q,q']}(\mathbf{r},\mathbf{Z},\mathbf{Z}'). \quad (89)$$

$S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta')$ is denoted by $S^{(\gamma)[q,q']}(\mathbf{r},\mathbf{Z},\mathbf{Z}')$ when only the shapes of the loops are important for the discussion. We notice that the present definition of $S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta')$ is more general than that used in Sec. III D of Ref. [5]. The latter definition was restricted to convolutions with algebraic bonds at both ends in order to preserve a tensorial structure, because, instead of property (B), we used the following property (B^*):

$$S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta') = A^{[q]}_{\mu_1,\ldots,\mu_q}(\mathbf{Z})A^{[q']}_{\nu_1,\ldots,\nu_{q'}}(\mathbf{Z}')S^{(\gamma)}_{\mu_1,\ldots,\mu_q\nu_1,\ldots,\nu_{q'}}(\mathbf{r})$$

where $A^{[q]}_{\mu_1,\ldots,\mu_q}(\mathbf{Z})$ and $A^{[q']}_{\nu_1,\ldots,\nu_{q'}}(\mathbf{Z}')$ are tensors of rank q and q', respectively, and $S^{(\gamma)}_{\mu_1,\ldots,\mu_q\nu_1,\ldots,\nu_{q'}}(\mathbf{r})$ decays as $1/r^{\gamma}$. In the present paper, we consider more general structures $S^{(\gamma)[q,q']}(\mathbf{r},\zeta,\zeta')$ so that property (B) is weaker than property (B^*). The advantage is that property (B) is valid in the presence as in the absence of magnetic field and it is sufficient for deriving the properties which are in common for both cases.

An argument similar to that given in Sec. III D of Ref. [5] shows that, before integration over loop shapes, the leading and subleading tails T of a diagram $\tilde{\Pi}$ have a structure which satisfies property (A), with $\gamma_T=\Sigma_{l=1}^{L}[P_l(q_l,q_l')+q_l+q_l']+Q_a+Q_b$, and property (B),

$$T(\mathbf{r}_a-\mathbf{r}_b,\chi_a,\chi_b) = S^{(\gamma_T)[Q_a+\Sigma_l q_l,Q_b+\Sigma_{l'}q_{l'}]}(\mathbf{r}_a-\mathbf{r}_b,\chi_a,\chi_b), \quad (90)$$

where l runs from 1 to L, and Q_a (Q_b) is the number of derivatives with respect to $\mathbf{r}_a$ $(\mathbf{r}_b)$ which are performed to obtain the subleading term T.

Since some tails arising from convolutions involving W bonds are *a priori* algebraic and prove to be short ranged after integration over loop shapes, we deal with convolutions $\mathcal{C}$ separately. According to the definition of the $\mathcal{K}^{[n_i m_i]}_{i,i+1}$'s, the convolutions $\mathcal{C}$ involve the algebraic tails of the inverse Fourier transforms of functions

$$\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}(\mathbf{k}\cdot\mathbf{X}_b)^{m_b}\tilde{\Pi}_{W_c}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b). \quad (91)$$

As shown in the following two sections, the conclusions are the same for the $\tilde{\Pi}_{W_c}$ diagrams in the absence as in the presence of $\mathbf{B}_0$. On the contrary, the discussion about the convolutions $\mathcal{C}$ is different whether $\mathbf{B}_0$ is switched on or not.

B. Tails arising from $\tilde{\Pi}_{W_c}$ diagrams

The contribution of a tail T coming from a

$$(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}(\mathbf{k}\cdot\mathbf{X}_b)^{m_b}\tilde{\Pi}_{W_c}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$$

decays as $1/r^{\gamma_T+n_a+m_b}$ before integration over the loop shapes $\mathbf{X}_a$ and $\mathbf{X}_b$. Since T is of parity $(-1)^{Q_a+\Sigma_l q_l}$ under $\mathbf{X}_a\to-\mathbf{X}_a$ and $(-1)^{Q_b+\Sigma_l q_l'}$ under $\mathbf{X}_b\to-\mathbf{X}_b$, the invariance of

$$\int D(\mathbf{X}_a)\int D(\mathbf{X}_b)(\mathbf{k}\cdot\mathbf{X}_a)^{n_a}(\mathbf{k}\cdot\mathbf{X}_b)^{m_b}\tilde{\Pi}_{W_c}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$$

under inversion implies that the only values $\Gamma_{W_c}(n_a,m_b)$ of $\gamma_T+n_a+m_b$ that survive after integration over the loop shapes $\mathbf{X}_a$ and $\mathbf{X}_b$ correspond to the even values of $n_a+Q_a+\Sigma_l q_l$ and $m_b+Q_b+\Sigma_l q_l'$, namely,

$$\begin{aligned}\Gamma(n_a,m_b;\tilde{\Pi}_{W_c}) = &\sum_{l=1}^{L}P_l(q_l,q_l')+\min\left\{n_a+Q_a+\sum_{l=1}^{L}q_l\right\}\\ &+\theta\left(\min\left\{n_a+Q_a+\sum_{l=1}^{L}q_l\right\}\right)\\ &+\min\left\{m_b+Q_b+\sum_{l=1}^{L}q_l'\right\}\\ &+\theta\left(\min\left\{m_b+Q_b+\sum_{l=1}^{L}q_l'\right\}\right)+2N,\end{aligned} \quad (92)$$

where $\min\{\cdots\}$ denotes the minimal value of $\{\cdots\}$ when the q_l's and q_l''s vary while Q_a takes any positive integer value (zero included). $\theta(n)$, which has been defined in Sec. VI B, ensures that we write the even values taken by $\{\cdots\}$. The point is to know both the minimal value taken by $\Sigma_{l=1}^{L}q_l$ and $\Sigma_{l=1}^{L}q_l'$ in order to determine the even values of $\Gamma(n_a,m_b;\tilde{\Pi}_{W_c})-\Sigma_{l=1}^{L}P_l(q_l,q_l')$, and the minimal odd and even values taken by $\Sigma_{l=1}^{L}P_l(q_l,q_l')$ when the q_l's and q_l''s vary.

First the minimal odd and even values taken by every $P(q,q')$ are derived in Appendix A. The important results are the following. For tails T of diagrams $\tilde{\Pi}_{W_c}$ with $L=1$, the algebraic asymptotic behavior of the single elementary algebraic tail $S^{(\gamma)[q,q']}$ does not involve any W bond or any convolution of W bonds, so that $q\geqslant 2$ and $q'\geqslant 2$. In this case, the first allowed value for $P(q,q')$ is

$$P_{\text{even,min}}(q,q';\tilde{\Pi}_{W_c},L=1)=2 \quad \text{with } q\geqslant 2 \text{ and } q'\geqslant 2, \quad (93)$$

which is realized by an F_{R6} bond. The minimal odd allowed value is given by convolutions involving only F_{R6} bonds, with the result

$$P_{\text{odd,min}}(q,q';\tilde{\Pi}_{W_c},L=1)=\begin{cases}5 & \text{if } \inf(q,q')=2\\ 3 & \text{if } \inf(q,q')\geqslant 3.\end{cases} \quad (94)$$

For tails T of diagrams $\tilde{\Pi}_{W_c}$ with $L\geqslant 2$, the various $S^{(\gamma)[q,q']}$'s may involve W bonds. Then, the minimal odd allowed value for $P(q,q')$ is

$$P_{\text{odd,min}}(q,q';\tilde{\Pi}_{W_c},L=2)=1 \quad \text{for any } (q,q'), \quad (95)$$

which is realized by convolutions where the nonanalytic terms arise only from W bonds. Moreover,

$$P_{\text{even, min}}(q,q';\tilde{\Pi}_{W_c},L=2)=\begin{cases}6 & \text{if } q=q'=1\\ 4 & \text{if } (q=1,\ q'\geqslant 2) \text{ or } (q\geqslant 2,\ q'=1)\\ 2 & \text{if } \inf(q,q')\geqslant 2.\end{cases} \quad (96)$$

When $\inf(q,q')\geqslant 2$, $P(q,q')=2$ is realized by an F_{R6} bond. $P(q,q')=6+2N$ and $P(q,q')=4+2N$ are realized by convolutions involving at least one F_{R6} bond. As a consequence, the minimal allowed value of $\Sigma_{l=1}^{L}P_l(q_l,q_l')$ for a $\tilde{\Pi}_{W_c}$ diagram is

$$\min\left\{\sum_{l=1}^{L}P_l(q_l,q_l')\right\}=\begin{cases}2 & \text{if } L=1 \text{ in which case } q_l\geqslant 2 \text{ and } q_l'\geqslant 2\\ L & \text{if } L\geqslant 2 \text{ in which case } q \text{ and } q' \text{ take any value.}\end{cases} \quad (97)$$

These minimal values are always realized.

Now, the point is to know the minimal odd value, $\min\{\delta_T^{\text{odd}}\}$, of $\delta_T\equiv\Sigma_{l=1}^{L}P_l(q_l,q_l')-\min\{\Sigma_{l=1}^{L}P_l(q_l,q_l')\}$. For a $\tilde{\Pi}_{W_c}$ diagram with $L=1$ [called case (IV) in the following as in Appendix C of Ref. [5]], $\delta_T=\Sigma_{l=1}^{L}P_l(q_l,q_l')-2$ and, according to Eq. (94), the discussion of $\min\{\delta_T^{\text{odd}}\}$ can be organized by inspection of the various cases $q=q'=2$ $(q=2,q'\geqslant 3)$ or $(q\geqslant 3,q'=2)$, and $\inf(q,q')\geqslant 3$. We get

$$\Gamma(n_a,m_b;\tilde{\Pi}_{W_c},L=1)-n_a-\theta(n_a)-m_b-\theta(m_b)-6=\begin{cases}0,1,\ldots & \text{if } n_a \text{ and } m_b \text{ are odd}\\ 0,2,3,\ldots & \text{in other cases.}\end{cases} \quad (98)$$

In the case $L\geqslant 2$, according to Eq. (96), $\min\{\Sigma_{l=1}^{L}P_l(q_l,q_l')\}=L$ and three cases are to be distinguished in order to determine the minimal odd value of $\delta_T=\Sigma_{l=1}^{L}P_l(q_l,q_l')-L$.

Case (I): $q_l=q_l'=1$ for all l. Then $\min\{\delta_T^{\text{odd}}\}=5$ and $\Sigma_{q_l=1}^{L}q_l=\Sigma_{q_l=1}{}^{L}q_l'=L$.

Case (II): there exists some l_0 such that $(q_{l_0}=1,\ q_{l_0}'\geqslant 2)$, while for all $l\neq l_0$, $(q_l=1,\ q_l'=1)$ or $(q_l=1,\ q_l'\geqslant 2)$ or $(q_l\geqslant 2,\ q_l'=1)$. Then $\min\{\delta_T^{\text{odd}}\}=3$ and $\Sigma_{q_l=1}^{L}q_l\geqslant L$ while $\Sigma_{q_l=1}^{L}q_l'\geqslant L+1$. The same is true when the roles of q_{l_0} and q_{l_0}' are exchanged. [We notice that in Ref. [5] there is a misprint in the definition of case (II), where $q_l=2$ must be replaced by $q_l\geqslant 2$.]

Case (III): there exists some l_0 such that $\inf(q_{l_0},q_{l_0}')\geqslant 2$. Then $\min\{\delta_T^{\text{odd}}\}=1$ and $\Sigma_{q_l=1}^{L}q_l\geqslant L+1$ and $\Sigma_{q_l=1}^{L}q_l'\geqslant L+1$.

Then the discussion of Appendix C of Ref. [5] can be resumed. (Contrarily to what was done in Ref. [5], the discussion is carried out for any n_a and m_b from the start, and the cases $n_a=0$ or $n_b=0$ are derived at the end.) The first odd value of δ_T is

$$\min\{\delta_T^{\text{odd}}\}=\begin{cases}5 & \text{in case (I)}\\ 3 & \text{in case (II)}\\ 1 & \text{in case (III)}\\ 3 & \text{if } \inf(q,q')=2 \text{ and } 1 \text{ if } \inf(q,q')\geqslant 3 \text{ in case (IV).}\end{cases} \quad (99)$$

By considering from the start the case where $n_a\neq 0$ and$m_b\neq 0$, we get the same final results as in Eq. (C40) of Ref. [5]. [The misprints in Appendix C of Ref. [5] and the mistake in Eq. (C37), which are given in Appendix B of the present paper, do not affect the final results.]

The results are

$$\mathcal{F}^{-1}\left[\int D(\mathbf{X}_a)D(\mathbf{X}_b)[\mathbf{k}\cdot\mathbf{X}_a(\tau)]^{n_a}\tilde{\Pi}_{W_c}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)\times[\mathbf{k}\cdot\mathbf{X}_b(\tau')]^{m_b}\right]\underset{r\to\infty}{\sim}\frac{1}{r^{na+m_b+\theta(n_a+m_b)+\delta(n_a,m_b)}}\ldots, \quad (100)$$

where, if n_a and m_b are odd, $\delta(n_a,m_b)=0,1,\ldots$ and $\delta(n_a,m_b)=0,2,3,\ldots$ in other cases. The result is the same as in the conclusion of Appendix B of Ref. [5] because the important property at stake is Eq. (99). As a consequence, up to misprints, the results derived in Appendix C of Ref. [5] and stated in Sec. III E of Ref. [5] are still valid,

$$\int D(\mathbf{X}_a)D(\mathbf{X}_b)\tilde{\Pi}_{W_c}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b) \underset{r\to\infty}{\sim} \frac{1}{r^6},\frac{1}{r^8},\frac{1}{r^9},\ldots, \tag{101}$$

where the brief notation in Eq. (101) means that there appear tails decaying as $1/r^6$, $1/r^8$, $1/r^\gamma$, with $\gamma\geqslant 9$. Moreover, the following property is valid for a $\tilde{\Pi}_{W_c}$ diagram in the absence as well as in the presence of $\mathbf{B}_0$,

$$\mathcal{F}^{-1}\left[\int D(\mathbf{X}_a)D(\mathbf{X}_b)[e^{i\mathbf{k}\cdot\mathbf{X}_a(\tau)}-1]\tilde{\Pi}_{W_c}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)\right] \underset{r\to\infty}{\sim} \frac{1}{r^8},\frac{1}{r^{10}},\frac{1}{r^{11}},\ldots \tag{102}$$

while

$$\mathcal{F}^{-1}\left[\int D(\mathbf{X}_a)D(\mathbf{X}_b)[e^{i\mathbf{k}\cdot\mathbf{X}_a(\tau)}-1]\tilde{\Pi}_{W_c}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b)\times[e^{-i\mathbf{k}\cdot\mathbf{X}_b(\tau')}-1]\right]_{r\to\infty} \sim \frac{1}{r^{10}},\frac{1}{r^{11}},\ldots . \tag{103}$$

As a comment, the discussion in Appendix A is analogous to that of Appendix B of Ref. [5]. Because of a mistake, the conclusions (B10) and (B11) of the latter Appendix turn out to be valid only in the presence of the magnetic field $\mathbf{B}_0$, whereas they must be modified when $\mathbf{B}_0=\mathbf{0}$, as displayed in Appendix C of the present paper. However, the conclusions (94) and (96), which are weaker than Eqs. (B10) and (B11), are valid whether $\mathbf{B}_0=\mathbf{0}$ or $\mathbf{B}_0\neq\mathbf{0}$ and they ensure that the $\Gamma(n_a,m_b;\tilde{\Pi}_{W_c})$'s are the same in the presence or in the absence of the magnetic field.

C. Tails arising from convolutions $\mathcal{C}$

Thanks to the study of the leading and subleading tails of $\tilde{\Pi}_{W_c}$, Eqs. (82) and (83) can be written more precisely as

$$\mathcal{K}_{a,1}^{[m_1]}(\mathbf{k})=O_{\text{anal}}^{(m_1+\theta(m_1))}(\mathbf{k})+O_{\text{anal}}^{(m_1+\theta(m_1)+2)}(\mathbf{k})+\cdots+S^{(m_1+\theta(m_1)+3)}(\mathbf{k})+\cdots, \tag{104}$$

where the next nonanalytic terms are of order $|\mathbf{k}|^{m_1+\theta(m_1)+5}$, $|\mathbf{k}|^{m_1+\theta(m_1)+6},\ldots,$ while

$$\mathcal{K}_{i,i+1}^{[n_i,m_i]}(\mathbf{k})=O_{\text{anal}}^{(n_i+m_{i+1}+\theta(n_i+m_{i+1}))}(\mathbf{k})+O_{\text{anal}}^{(n_i+m_{i+1}+\theta(n_i+m_{i+1})+2)}(\mathbf{k})+\cdots+S^{(n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+3)}(\mathbf{k})+\cdots, \tag{105}$$

with $\theta(n_i)+\theta(m_{i+1})\geqslant\theta(n_i+m_{i+1})$. Nonanalytic terms appear at every order $|\mathbf{k}|^{\gamma-3}$ with $\gamma-3\geqslant n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+4$ if n_i and m_{i+1} are odd, and $\gamma-3\geqslant n_i+m_{i+1}+\theta(n_i)+\theta(m_{i+1})+5$ in other cases. The discussion in Sec. III F of Ref. [5] can be resumed by replacing Eqs. (3.34) and (3.35), which are valid only when $\mathbf{B}_0=\mathbf{0}$, by Eqs. (104) and (105) respectively. The analysis of the nonanalytic terms is similar to that performed in Sec. VI C of the present paper. $\mathcal{C}_I(\mathbf{k},\{m_i\},\{n_i\})$ contains two kinds of nonanalytic terms. On one hand, the terms $[\mathbf{k}]_z^{2n}/(\mathbf{k}^2)^p$, which arise from the breaking of rotational invariance by $\mathbf{B}_0$, are of order $D_{\mathcal{C}_I},D_{\mathcal{C}_I}+2,D_{\mathcal{C}_I}+4,\ldots$ in $|\mathbf{k}|$, where $D_{\mathcal{C}_I}$ is given by the dimensional analysis of Sec. VI B. On the other hand, the nonanalytic terms involving at least one $\mathcal{K}$ are of order $D_{\mathcal{C}_I}+3$, $D_{\mathcal{C}_I}+5$, $D_{\mathcal{C}_I}+6,\ldots$ as in the discussion of Sec. III F of Ref. [5]. According to Eq. (84), $D_{\mathcal{C}_{I\min}}=2$ and

$$\int D(\mathbf{X}_a)D(\mathbf{X}_b)\mathcal{C}_{\mathbf{B}_0}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b) \underset{r\to\infty}{\sim} \frac{1}{r^5},\frac{1}{r^7},\frac{1}{r^8},\ldots \tag{106}$$

whereas

$$\int D(\mathbf{X}_a)D(\mathbf{X}_b)\mathcal{C}_{\mathbf{B}_0=\mathbf{0}}(\mathbf{r},\mathbf{X}_a,\mathbf{X}_b) \underset{r\to\infty}{\sim} \frac{1}{r^8},\frac{1}{r^{10}},\frac{1}{r^{11}},\ldots . \tag{107}$$

According to the dimensional analysis already performed in Appendix D of Ref. [5], the structure of the nonanalytic terms in $\int D(\mathbf{X}_a)D(\mathbf{X}_b)[\exp\{i\mathbf{k}\cdot\mathbf{X}_a(\tau)\}-1]\mathcal{C}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)$ is the same as for $\mathcal{C}_I(\mathbf{k},\{m_i\},\{n_i\})$, with $D_{\mathcal{C}_I}$ replaced by

$$\tilde{D}_{\mathcal{C}_I}=-2I+[1+m_1+\theta(1+m_1)]+n_I+\theta(n_I)+\sum_{i=1}^{I-1}[n_i+m_{i+1}+\theta(n_i+m_{i+1})]. \tag{108}$$

The minimal value of $\tilde{D}_{\mathcal{C}_I}$ is also 2 and

$$\mathcal{F}^{-1}\left[\int D(\mathbf{X}_a)D(\mathbf{X}_b)[e^{i\mathbf{k}\cdot\mathbf{X}_a(\tau)}-1]\mathcal{C}_{\mathbf{B}_0}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)\right] \underset{r\to\infty}{\sim} \frac{1}{r^5},\frac{1}{r^7},\frac{1}{r^8},\ldots \tag{109}$$

whereas

$$\mathcal{F}^{-1}\left[\int D(\mathbf{X}_a)D(\mathbf{X}_b)[e^{i\mathbf{k}\cdot\mathbf{X}_a(\tau)}-1]\mathcal{C}_{\mathbf{B}_0=\mathbf{0}}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)\right] \underset{r\to\infty}{\sim} \frac{1}{r^8},\frac{1}{r^{10}},\frac{1}{r^{11}},\ldots . \tag{110}$$

In the case of $\int D(\mathbf{X}_a)D(\mathbf{X}_b)[\exp\{i\mathbf{k}\cdot\mathbf{X}_a(\tau)\}-1]\mathcal{C}(\mathbf{k},\mathbf{X}_a,\mathbf{X}_b)[\exp\{-i\mathbf{k}\cdot\mathbf{X}_b(\tau')\}-1]$, $D_{\mathcal{C}_I}$ is replaced by

$$\tilde{\tilde{D}}_{\mathcal{C}_I} = -2I + [1 + m_1 + \theta(1+m_1)] + [1 + n_I + \theta(1+n_I)] + \sum_{i=1}^{I-1} [n_i + m_{i+1} + \theta(n_i + m_{i+1})] \tag{111}$$

but the nonanalytic terms involving at least one $\mathcal{K}$ appear only at the order $\tilde{\tilde{D}}_{\mathcal{C}_I}+5$. The minimal value for $\tilde{\tilde{D}}_{\mathcal{C}_I}$ is equal to 2 also. Thus

$$\mathcal{F}^{-1}\left[\int D(\mathbf{X}_a) D(\mathbf{X}_b) [e^{i\mathbf{k}\cdot\mathbf{X}_a(\tau)} - 1] \mathcal{C}_{\mathbf{B}_0}(\mathbf{k}, \mathbf{X}_a, \mathbf{X}_b) \times [e^{-i\mathbf{k}\cdot\mathbf{X}_a(\tau')} - 1] \right]_{r\to\infty} \sim \frac{1}{r^5}, \frac{1}{r^7}, \frac{1}{r^9}, \frac{1}{r^{10}}, \ldots \tag{112}$$

whereas

$$\mathcal{F}^{-1}\left(\int D(\mathbf{X}_a) D(\mathbf{X}_b) [e^{i\mathbf{k}\cdot\mathbf{X}_a(\tau)} - 1] \mathcal{C}_{\mathbf{B}_0=\mathbf{0}}(\mathbf{k}, \mathbf{X}_a, \mathbf{X}_b) \times [e^{-i\mathbf{k}\cdot\mathbf{X}_a(\tau')} - 1] \right)_{r\to\infty} \sim \frac{1}{r^{10}}, \frac{1}{r^{11}}, \ldots . \tag{113}$$

The present results when $\mathbf{B}_0 = \mathbf{0}$ are more precise than those given in Ref. [5]. (We notice that some misprints in Appendix D of Ref. [5] do not affect the results given in the latter reference.) According to Eqs. (107), (110), and (113), the tails $1/r^5$, $1/r^7$, and $1/r^9$ disappear when $\mathbf{B}_0 = \mathbf{0}$, because they come only from the nonanalyticities due to the breaking of rotational invariance in the presence of $\mathbf{B}_0$, whereas tails $1/r^{10+N}$, with $N \geqslant 0$, come from both breaking of rotational invariance and singularities in the $\mathcal{K}$'s.

VIII. LEADING AND SUBLEADING ALGEBRAIC TAILS OF VARIOUS CORRELATIONS

A. Interplay with the ''Debye dressing''

First, we exhibit a property of the bond F^{cc} when charges are summed over. In the following the ''Debye'' polarization cloud of loops around a loop $\mathcal{L}_a$ is defined as

$$\Sigma_D(\mathbf{R}_a - \mathbf{R}_1, \chi_1; \chi_a) = \delta_{\chi_a,\chi_1}\delta(\mathbf{R}_a - \mathbf{R}_1) + \rho(\chi_1) F^{cc}(\mathcal{L}_a, \mathcal{L}_1). \tag{114}$$

In Σ_D the variable after '';'' always denotes a root point. (This notation is slightly different from that of Ref. [5] and is more precise.) The property

$$\int d\chi_a e_{\alpha_a} p_a \rho(\chi_a) \Sigma_D(\mathbf{k}, \chi_1; \chi_a) = e_{\alpha_1} p_1 \rho(\chi_1) \frac{\mathbf{k}^2}{\kappa^2 + \mathbf{k}^2} = \frac{e_{\alpha_1} p_1 \rho(\chi_1)}{\kappa^2} \mathbf{k}^2 + O(|\mathbf{k}|^4) \tag{115}$$

implies that, if a diagram Π behaves as $1/r^n$ and may be convoluted with F^{cc} bonds, then the contribution of $\Sigma_D^* \Pi$ to $\Sigma_\alpha e_\alpha \rho^{(2)\,T}_{\alpha\gamma}(\mathbf{r})$ falls off at least as $1/r^{n+2}$ and the contribution from $\Sigma_D * \Pi * \Sigma_D$ to $\Sigma_{\alpha,\gamma} e_\alpha e_\gamma \rho^{(2)\,T}_{\alpha\gamma}(\mathbf{r})$ decays at least as $1/r^{n+4}$. More precisely, the tails $1/r^{n+2}$ and $1/r^{n+4}$ do exist only if the $\mathbf{k}^2$ term arising from Eq. (115) does not cancel the $1/\mathbf{k}^2$ singularity of the Coulomb potential; otherwise, the leading algebraic tails are replaced by short-ranged behaviors.

The previous mechanism for a cascade of power laws can be worked out as follows. First we reorganize the diagrams in order to produce integral relations in which Σ_D appears explicitly. For that purpose, we introduce the following definitions. We call a ''Coulomb-root'' point a root point $\mathcal{L}_a$ that is involved either in one and only one bond $F^{cc}(\mathcal{L}_a, \mathcal{P}_i)$ or $F^{cm}(\mathcal{L}_a, \mathcal{P}_i)$. On the contrary, a non-Coulomb-root point $\mathcal{L}_a$ is involved either in one bond $F_R(\mathcal{L}_a, \mathcal{P}_i)$ or $F^{mc}(\mathcal{L}_a, \mathcal{P}_i)$ or in at least two bonds, whatever they are. Let $h^{n-}(\mathcal{L}_a, \mathcal{L}_b)$ be the sum of the Π diagrams where $\mathcal{L}_a$ is a non-Coulomb-root point, whereas $\mathcal{L}_b$ is of any kind (Coulomb-root or non-Coulomb-root point). $h^{nn}(\mathcal{L}_a, \mathcal{L}_b)$ is defined in a similar way. With these definitions, the excluded-convolution rules lead to the left-dressing relation

$$h = F^{cc} + F^{cm} + \Sigma_D * h^{n-} + F^{cm} \rho * h \tag{116}$$

as well as to the right-dressing relation

$$h = F^{cc} + F^{mc} + h^{-n} * \Sigma_D + h * \rho F^{mc}, \tag{117}$$

where the definition of $h^{-n}(\mathcal{L}_a, \mathcal{L}_b)$ is obtained from that of $h^{n-}(\mathcal{L}_a, \mathcal{L}_b)$ by exchanging the roles of $\mathcal{L}_a$ and $\mathcal{L}_b$. These relations are convolutions for the loop-position variable while the internal degrees of freedom of the intermediate loop are integrated over. In these short notations, we use the convention that ρ is the density of the intermediate point of the convolution and that Σ_D is the Debye polarization cloud around the root point of the convolution (as detailed in Sec. IV C of [5]). There are two extra integral relations,

$$h^{-n} = F^{cm} + \Sigma_D * h^{nn} + F^{cm} \rho * h^{-n} \tag{118}$$

and

$$h^{n-} = F^{mc} + h^{nn} * \Sigma_D + h^{n-} * \rho F^{mc}. \tag{119}$$

By using the above relations repeatedly, we decompose h as a sum of five terms which are convenient to discuss the leading and subleading tails of various correlations because they exhibit dressings by Σ_D and ρF^{mc}. The decomposition is introduced in Sec. IV D of Ref. [1]. The first one, $h_{(A)}$, decays faster than any inverse power law of the distance and the other ones read

$$h_{(B)} \equiv \Sigma_D * h^{nn} * \Sigma_D, \tag{120a}$$

$$h_{(C)} \equiv \Sigma_D * h^{n-} * \rho F^{mc}, \tag{120b}$$

$$h_{(D)} \equiv F^{cm} \rho * h^{-n} * \Sigma_D, \tag{120c}$$

$$h_{(E)} \equiv F^{cm} \rho * h * \rho F^{mc}. \tag{120d}$$

The leading and subleading algebraic tails of the particle-particle correlations at large distances are derived from the detailed study of the asymptotic behaviors (101) and (106) of diagrams $\tilde{\Pi}_{W_c}$ and convolutions $\mathcal{C}$, respectively. We find that

$$\rho^{(2)T}_{\alpha\gamma}|_{\mathbf{B}_0} \underset{r\to\infty}{\sim} \frac{1}{r^5}, \frac{1}{r^6}, \ldots . \tag{121}$$

Inspection of the more refined results (101)–(103), (106), (109), and (112) shows the following important results which are valid in the presence as well as in the absence of B_0 (apart from the first one which is only relevant to the the case $\mathbf{B}_0 \neq \mathbf{0}$).

(I) The $1/r^5$ tail comes from $h_{(B)}+h_{(C)}+h_{(D)}+h_{(E)}$.

(II) The $1/r^6$ tail originates only from $h_{(B)}$.

(III) The $1/r^8$ tail arises from $h_{(B)}+h_{(C)}+h_{(D)}$.

(IV) The $1/r^{10}$ tail comes from $h_{(B)}+h_{(C)}+h_{(D)}+h_{(E)}$.

The cascade of power laws may now be discussed thanks to the above remark about the origin of the subleading tails in terms of the contributions $h_{(B)}$, $h_{(C)}$, $h_{(D)}$, and $h_{(E)}$. Indeed, according to Eq. (115), for $\mathbf{B}_0=\mathbf{0}$ or $\mathbf{B}_0\neq\mathbf{0}$, the terms contributing to the $1/r^6$ tail of $\rho^{(2)T}_{\alpha\gamma}$ become $1/r^8$ ($1/r^{10}$) tails in $\Sigma_\gamma e_\gamma \rho^{(2)T}_{\alpha\gamma}$ ($\Sigma_{\alpha,\gamma} e_\alpha e_\gamma \rho^{(2)T}_{\alpha\gamma}$) or decay faster; the terms contributing to the $1/r^8$ tail of $\rho^{(2)T}_{\alpha\gamma}$ become at least $1/r^{10}$ tails when charges of both species are summed over.

Consequently, the tails of the particle-charge and charge-charge correlations in the presence of $\mathbf{B}_0$ are

$$\sum_\gamma e_\gamma \rho^{(2)T}_{\alpha\gamma}|_{\mathbf{B}_0} \underset{r\to\infty}{\sim} \frac{1}{r^5}, \frac{1}{r^7}, \frac{1}{r^8}, \ldots, \tag{122}$$

$$\sum_{\alpha,\gamma} e_\alpha e_\gamma \rho^{(2)T}_{\alpha\gamma}|_{\mathbf{B}_0} \underset{r\to\infty}{\sim} \frac{1}{r^5}, \frac{1}{r^7}, \frac{1}{r^9}, \frac{1}{r^{10}}, \ldots . \tag{123}$$

When $\mathbf{B}_0=\mathbf{0}$, the $1/r^5$ and $1/r^7$ tails, which arise from the convolutions $\mathcal{C}$, disappear in $\rho^{(2)T}_{\alpha\gamma}$ according to Eq. (107). Moreover, inspection of Eqs. (101)–(103), (107), (110), and (113) shows that the $1/r^9$ tail of $\rho^{(2)T}_{\alpha\gamma}$ comes only from $h_{(B)}$ when $\mathbf{B}_0=\mathbf{0}$ and, according to Eq. (115), it disappears as soon as charges are summed over (because the order of the possible singularity in Fourier space is increased by a term proportional to $|\mathbf{k}|^2$). As a consequence,

$$\rho^{(2)T}_{\alpha\gamma}|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{1}{r^6}, \frac{1}{r^8}, \frac{1}{r^9}, \ldots, \tag{124}$$

$$\sum_\gamma e_\gamma \rho^{(2)T}_{\alpha\gamma}|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{1}{r^8}, \frac{1}{r^{10}}, \frac{1}{r^{11}}, \ldots, \tag{125}$$

$$\sum_{\alpha,\gamma} e_\alpha e_\gamma \rho^{(2)T}_{\alpha\gamma}|_{\mathbf{B}_0=\mathbf{0}} \underset{r\to\infty}{\sim} \frac{1}{r^{10}}, \frac{1}{r^{11}}, \ldots . \tag{126}$$

B. Diagrammatic structure of leading tails

In this section we only consider the leading tails of the particle-particle, particle-charge, and charge-charge correlations. We show that they can be expressed only in terms of h^{nn} with various dressings that involve Σ_D or/and ρF^{mc}.

1. Basic properties

The derivation relies on two kinds of ingredients. First, we use repeatedly dressing relations that are valid in the presence as well as in the absence of $\mathbf{B}_0$. Some have already been given in Eqs. (116)–(119); the other ones are

$$F^{cc}*\rho F^{mc}(\mathbf{k},\chi_2,\chi_b)=O(|\mathbf{k}|^2) \tag{127}$$

and

$$F^{mc}*\rho F^{mc}(\mathbf{k},\chi_2,\chi_b)=O(|\mathbf{k}|^2). \tag{128}$$

Second, the detailed survey of the decay of diagrams has shown that, in the presence of $\mathbf{B}_0$, any diagram decays at least as $1/r^5$ after integration over the shapes $\mathbf{X}$ of the root points,

$$\int D(\mathbf{X}_a)\rho(\chi_a)\int D(\mathbf{X}_b)\rho(\chi_b)\Pi_{\mathbf{B}_0}(\mathbf{r},\chi_a,\chi_b) \underset{r\to\infty}{\sim} \frac{1}{r^5} \tag{129}$$

whereas, in the absence of $\mathbf{B}_0$, Eqs. (103) and (113) imply that

$$\mathcal{F}^{-1}\Big[\int D(\mathbf{X}_a)\rho(\chi_a)\int D(\mathbf{X}_b)\rho(\chi_b) \times[e^{i\mathbf{k}\cdot\mathbf{X}_a(\tau)}-1]\Pi_{\mathbf{B}_0=\mathbf{0}}(\mathbf{k},\chi_a,\chi_b)\Big] \underset{r\to\infty}{\sim} \frac{1}{r^8} \tag{130}$$

and

$$\mathcal{F}^{-1}\Big[\int D(\mathbf{X}_a)\rho(\chi_a)\int D(\mathbf{X}_b)\rho(\chi_b)[e^{-i\mathbf{k}\cdot\mathbf{X}_a(\tau)}-1] \times[e^{i\mathbf{k}\cdot\mathbf{X}_b(\tau')}-1]\Pi_{\mathbf{B}_0=\mathbf{0}}(\mathbf{k},\chi_a,\chi_b)\Big] \underset{r\to\infty}{\sim} \frac{1}{r^{10}}. \tag{131}$$

Third, the Debye screening described by Eq. (115) will play a role when charges are summed over.

2. In the presence of B_0

The analysis based on the previous properties shows that the leading $1/r^5$ tail of the particle-particle correlation $\rho^{(2)T}_{\alpha\gamma}$ comes only from

$$\Sigma^*_D * h^{nn} * \Sigma^*_D. \tag{132}$$

In Eq. (132) we have set

$$\Sigma^*_D(\mathbf{r},\chi_2;\chi_b)\equiv\Sigma_D(\mathbf{k},\chi_2;\chi_b)+\rho(\chi_2)F^{mc}(\mathbf{k},\chi_2,\chi_b), \tag{133}$$

where the variable after ";" is a Coulomb-root point for Σ_D as well as for F^{mc} and the superscripts m and c are associated with the internal point χ_2 and the root point χ_b, respectively.

The $1/r^5$ asymptotic behavior of the particle-charge correlation $\Sigma_\gamma e_\gamma \rho^{(2)T}_{\alpha\gamma}$ may originate only from the $1/r^5$ tail of the particle-particle correlation. According to Eq. (115), if

$\int D(\mathbf{X}_a)\rho(\chi_a)\int D(\mathbf{X}_b)\rho(\chi_b)[\Pi * \Sigma_D](\mathbf{r},\chi_a,\chi_b)$ decreases as $1/r^n$, then, after summation over e_γ the $1/r^n$ falloff turns into a slower decay, at least $1/r^{n+2}$. Therefore, in $\Sigma_D^* * h^{nn} * \Sigma_D^*$ only the part

$$\Sigma_D^* * h^{nn} * \rho F^{mc} \tag{134}$$

does contribute to the $1/r^5$ tail of the particle-charge correlation. For the same reason, the $1/r^5$ asymptotic behavior of the charge-charge correlation arises only from

$$F^{cm}\rho * h^{nn} * \rho F^{mc}. \tag{135}$$

We notice that, as charges are summed over, the $1/r^5$ tails, all of which originate from the same diagrammatic structure (132), involve in fact fewer and fewer contributions.

3. In the absence of B_0

As already mentioned in Sec. VIII A, the leading $1/r^6$ decay of the particle-particle correlation $\rho_{\alpha\gamma}^{(2)T}$ is due only to $h_{(B)}$, namely, to

$$\Sigma_D * h^{nn} * \Sigma_D. \tag{136}$$

On the contrary, the $1/r^8$ subleading tail of $\rho_{\alpha\gamma}^{(2)T}$ comes from $h_{(B)}+h_{(C)}+h_{(D)}$. After summation over the charge e_γ, the $1/r^8$ tail coming from $h_{(D)}$ turns into a $1/r^{10}$ decay. By using the basic properties (127), (128), and (130) together with the fact that any diagram decays at least as $1/r^6$ after integration over the loop shapes, the $1/r^8$ tail in $h_{(C)}$ is shown to originate only from the part $\Sigma_D * h^{nn} * [\rho F^{mc} + \rho F^{cc} * \rho F^{mc}]$. Eventually, the $1/r^8$ behavior of the particle-charge correlation reduces to the asymptotic decay of

$$\Sigma_D * h^{nn} * \Sigma_D^{**}. \tag{137}$$

In Eq. (137) $\Sigma_D^{**}(\mathbf{k},\chi_2;\chi_b)$ is defined as

$$\Sigma_D^{**} \equiv \Sigma_D + \rho F^{mc} + \rho F^{cc} * \rho F^{mc} = \Sigma_D * [\delta + \rho F^{mc}], \tag{138}$$

where the variable after ";" in $\Sigma_D^{**}(\mathbf{k},\chi_2;\chi_b)$ is a Coulomb-root point as in the definition of Σ_D. The $1/r^{10}$ subleading tail of $\rho_{\alpha\gamma}^{(2)T}$ originates from $h_{(B)}+h_{(C)}+h_{(D)}+h_{(E)}$. By using the same arguments as above, one shows that the $1/r^{10}$ tail of $h_{(C)}$ comes in fact only from $\Sigma_D * h^{nn} * \Sigma_D * \rho F^{mc}$, that of $h_{(D)}$ from $F^{cm}\rho * \Sigma_D * h^{nn} * \Sigma_D$, and that of $h_{(E)}$ from $F^{cm}\rho * \Sigma_D * h^{nn} * \Sigma_D * \rho F^{mc}$. Eventually, the $1/r^{10}$ tail of the charge-charge correlation originates only from

$$\Sigma_D^{**} * h^{nn} * \Sigma_D^{**}. \tag{139}$$

As a final remark, we compare the formulas in both cases, $\mathbf{B}_0=\mathbf{0}$ and $\mathbf{B}_0\neq\mathbf{0}$. On one hand, in the presence of $\mathbf{B}_0$, the Debye screening relation (115) makes Σ_D disappear in the $1/r^5$ tail as more charges are summed over. On the other hand, in the absence of $\mathbf{B}_0$, Σ_D is responsible for the cascade of power laws in the leading tails and it remains in the diagrams that do contribute to the coefficients of the asymptotic behaviors. As a consequence, the diagrammatic structure of the latter ones is more and more complex as charges are summed over.

C. Induced charge

1. Basic formulas

First, we exhibit the expression of the internal screening rule (1) in terms of the loop Ursell function h and its "Debye" approximation F^{cc}. According to Eqs. (33), (34), (71), and Eq. (4.9) of Ref. [1], Eq. (1) reads

$$0 = \int d\mathbf{r} \sum_\gamma e_\gamma S_{\alpha\gamma}(\mathbf{r}) = \sum_{p_a=1}^{\infty} \int D(\mathbf{X}_a) p_a \rho(\chi_a) \times G_{\{h-F^{cc}\}}(\mathbf{k}=\mathbf{0},\chi_a), \tag{140}$$

where

$$G_f(\mathbf{k},\chi) \equiv \int d\chi_b\, \rho(\chi_b) e_{\alpha_b} \int_0^{p_b} d\tau\, e^{-i\mathbf{k}\cdot\mathbf{X}_b(\tau)} f(\mathbf{k},\chi;\chi_b). \tag{141}$$

The induced charge $\Sigma_\gamma e_\gamma \rho_\gamma^{\mathrm{ind}}(r;\delta q)$ in the presence of an infinitesimal external point charge δq located at $\mathbf{r}=\mathbf{0}$ can be derived in two different ways. First, it may be obtained by linearizing the result for the particle-charge correlation $\Sigma_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(r)$ with respect to the charge e_α. Indeed, a quite general statement is that

$$\sum_\gamma e_\gamma \rho_\gamma^{\mathrm{ind}}(r;e_\alpha,\rho_\alpha=0) = \lim_{\rho_\alpha\to 0} \frac{\sum_\gamma e_\gamma \rho_{\alpha\gamma}^{(2)T}(r)}{\rho_\alpha}. \tag{142}$$

This relation states that the charge density induced by one charge e_α different from those in the plasma can be retrieved from the particle-charge correlation in the limit where one species α becomes more and more dilute, so that it disappears from the plasma. In order to obtain the response to an infinitesimal charge, one must linearize the right-hand side of Eq. (142) with respect to e_α.

The induced charge may also be calculated directly from the linear response theory, valid for any distribution $\delta q(\mathbf{r})$. According to Sec. IV E of Ref. [5], the structure of the latter formula is different from the expression of the particle-charge correlation. It reads

$$\frac{\sum_\alpha e_\alpha \rho_\alpha^{\mathrm{ind}}(\mathbf{k};\delta q)}{\delta q(\mathbf{k})} = -\frac{\kappa^2(\mathbf{k})}{\kappa^2+\mathbf{k}^2} - \frac{4\pi\beta}{\mathbf{k}^2}\int d\chi_a p_a e_{\alpha_a} \rho(\chi_a) G_{\{h-F^{cc}\}}(\mathbf{k},\chi_a), \tag{143}$$

where we have set

$$\kappa^2(\mathbf{k}) \equiv 4\pi\beta \int d\chi p^2 e_\alpha^2 \rho(\chi) \int_0^p \frac{d\tau}{p} e^{i\mathbf{k}\cdot\mathbf{X}(\tau)}. \tag{144}$$

According to the definition (34) of κ,

$$\kappa^2(\mathbf{k})=\kappa^2[1+A^{(2)}_{\rm anal}(\mathbf{k})]+O(|\mathbf{k}|^4), \tag{145}$$

where $A^{(2)}_{\rm anal}(\mathbf{k})$ is a term of order $|\mathbf{k}|^2$ which is analytic in the components of $\mathbf{k}$. [The next term is of order $|\mathbf{k}|^4$ because $\rho(\chi)$ is invariant under inversion of $\mathbf{X}$.] When $\mathbf{B}_0=\mathbf{0}$, $A^{(2)}_{\rm anal}(\mathbf{k})$ is exactly proportional to $\mathbf{k}^2$, whereas, when $\mathbf{B}_0\neq\mathbf{0}$, $A^{(2)}_{\rm anal}(\mathbf{k})$ may be written as the sum of a $\mathbf{k}^2$ term and a $[\mathbf{k}]^2_z$ term. Moreover, $A^{(2)}_{\rm anal}(\mathbf{k})$ starts at order zero in loop density.

2. Perfect external screening

First, we recall the mechanism in the classical case. The classical Ursell function $h^{\rm cl,reg}_{\alpha\gamma}(\mathbf{r}_a,\mathbf{r}_b)$ can be decomposed as

$$h^{\rm cl,reg}_{\alpha\gamma}=F^{cc}_{D,\alpha\gamma}+\sum_{\alpha_1,\alpha_2}\Sigma^{\rm cl}_D(\alpha_1;\alpha)*h^{nn\ {\rm cl,reg}}_{\alpha_1\alpha_2}*\Sigma^{\rm cl}_D(\alpha_2;\gamma), \tag{146}$$

where $F^{cc}_{D,\alpha\gamma}\equiv-\beta e_\alpha e_\gamma\phi_D$ and ϕ_D is the Debye potential $\phi_D=\exp(-\kappa_D r)/r$ with $\kappa_D\equiv\sqrt{4\pi\beta\Sigma_\alpha\rho_\alpha e_\alpha^2}$. $\Sigma^{\rm cl}_D(\mathbf{r}_a-\mathbf{r}_b,\gamma;\alpha)\equiv\delta_{\alpha,\gamma}\delta(\mathbf{r}_a-\mathbf{r}_b)+\rho_\gamma F^{cc}_{D,\alpha\gamma}$ and $h^{nn\ {\rm cl,reg}}_{\alpha\gamma}$ is the sum of the diagrams with non-Coulomb-root points that are built with the bonds $F^{cc}_{D,\alpha\gamma}$ and $F^{\rm cl,reg}_{R,\alpha\gamma}\equiv\exp[F^{cc}_{D,\alpha\gamma}-\beta v_{\rm SR}]-1-F^{cc}_{D,\alpha\gamma}$, where $v_{\rm SR}$ is a repulsive short-ranged potential that prevents the collapse of opposite charges ($v_{\rm SR}$ was omitted in Sec. IV A of Ref. [5]). Thus $h^{nn\ {\rm cl,reg}}$ decays faster than any inverse power law and its Fourier transform is analytic. As shown in Ref. [5], since

$$\sum_\gamma e_\gamma\Sigma^{\rm cl}_D(\mathbf{k},\gamma;\alpha)=\frac{e_\alpha}{\kappa_D^2}\mathbf{k}^2+O(|\mathbf{k}|^4) \tag{147}$$

and, according to Eq. (146), the internal screening rules (1) and (2) are satisfied by $\Sigma_\gamma e_\gamma S^{\rm cl,reg}_{\alpha\gamma}$ as well as by its Debye approximation $\Sigma_\gamma e_\gamma\Sigma^{\rm cl}_D$. Moreover, the first term in the small-$\mathbf{k}$ expansion of the classical charge-charge correlation $C^{\rm cl,reg}(\mathbf{k})\equiv\Sigma_{\alpha,\gamma}e_\alpha e_\gamma\rho^{(2)\,T{\rm cl,reg}}_{\alpha\gamma}(\mathbf{k})+\Sigma_\alpha e_\alpha\rho_\alpha^2$ is equal to its value in the Debye approximation. Therefore both charge-charge correlations (the exact and Debye expressions) satisfy the Stillinger-Lovett sum rule

$$C^{\rm cl,reg}(\mathbf{k})\underset{|\mathbf{k}|\to0}{\sim}\frac{\mathbf{k}^2}{4\pi\beta}. \tag{148}$$

On the other hand, according to the linear response relation in the classical regime,

$$\frac{\sum_\alpha e_\alpha\rho^{\rm ind,cl}_\alpha(\mathbf{k};\delta q)}{\delta q(\mathbf{k})}=-\beta C^{\rm cl,reg}(\mathbf{k})v_C(\mathbf{k}). \tag{149}$$

Thus the rule (148) ensures that an infinitesimal external charge distribution is completely screened by the medium: $\Sigma_\gamma e_\gamma\rho^{\rm ind,cl}_\gamma(\mathbf{k}=0)=-\delta q(\mathbf{k}=\mathbf{0})$.

In the quantum case, according to Eqs. (140), (143), and (144), when the Ursell function h is approximated by the sole bond F^{cc}, it happens to satisfy both the internal and perfect external screening conditions (1), (2), and (3). In other words, F^{cc} saturates the basic screening sum rules, as F^{cc}_D does in the classical case. Thus, according to Eqs. (140) and (143), the proof of the internal and perfect external screening amounts to showing that $h-F^{cc}$ gives a contribution of order greater than $|\mathbf{k}|$ to $\Sigma_{p_a}\int D(\mathbf{X}_a)p_a\rho(\chi_a)G_{\{h-F^{cc}\}}(\mathbf{k},\chi_a)$ and greater than $|\mathbf{k}|^2$ to $\int d\chi_a p_a e_{\alpha_a}\rho(\chi_a)G_{\{h-F^{cc}\}}(\mathbf{k},\chi_a)$.

First, we notice that, though the small-$\mathbf{k}$ expansions of G_{Σ_D} and $G_{\rho F^{mc}}$ start at the order $|\mathbf{k}|$ separately, their sum $G_{\Sigma^*_D}$ starts at the order $|\mathbf{k}|^2$,

$$G_{\Sigma^*_D}(\mathbf{k},\chi_2)=\rho(\chi_2)p_2e_{\alpha_2}\left[1-\frac{\kappa^2(\mathbf{k})}{\kappa^2+\mathbf{k}^2}\right]\int_0^{p_2}\frac{d\tau}{p_2}e^{-i\mathbf{k}\cdot\mathbf{X}_2(\tau)}$$
$$=p_2e_{\alpha_2}\left[\frac{\mathbf{k}^2}{\kappa^2}-A^{(2)}_{\rm anal}(\mathbf{k})+O(|\mathbf{k}|^3)\right]. \tag{150}$$

Therefore we use the dressing relations of Sec. VIII A repeatedly in order to make Σ^*_D appear on the right side of the expression of $h-F^{cc}$. Then, as already done in the case where $\mathbf{B}_0=\mathbf{0}$ in Sec. IV E of Ref. [5], $h-F^{cc}$ is written as the sum of three contributions: $h_{(A^*)}$ which decays faster than any inverse power law of the distance,

$$h_{(A^*)}\equiv\Sigma_D*F^{mc}+F^{cm}*\Sigma^{**}_D+\{\Sigma_D+F^{cm}\rho\}*F^{mc}*\rho F^{mc}, \tag{151}$$

$$h_{(B^*)}\equiv\Sigma_D*\{h^{nn}*\Sigma^{**}_D+h^{n-}*\rho F^{mc}*\rho F^{mc}\}, \tag{152}$$

and

$$h_{(C^*)}\equiv F^{cm}\rho*\{h^{-n}*\Sigma^{**}_D+h*\rho F^{mc}*\rho F^{mc}\}. \tag{153}$$

In fact, there appears a right dressing not only by Σ^*_D but also by $\rho F^{cc}*\rho F^{mc}$, so that Σ^{**}_D shows up again together with another right dressing by $\rho F^{mc}*\rho F^{mc}$. Since

$$G_{\{\rho F^{cc}*\rho F^{mc}\}}(\mathbf{k},\chi_2)=\rho(\chi_2)p_2e_{\alpha_2}\frac{\kappa^2(\mathbf{k})[\kappa^2(\mathbf{k})-\kappa^2]}{(\kappa^2+\mathbf{k}^2)^2}$$
$$=\rho(\chi_2)p_2e_{\alpha_2}A^{(2)}_{\rm anal}(\mathbf{k})+O(|\mathbf{k}|^4) \tag{154}$$

the small-$\mathbf{k}$ expansion of $G_{\Sigma^{**}_D}$ starts by a $\mathbf{k}^2$ term,

$$G_{\Sigma^{**}_D}(\mathbf{k},\chi_2)=\rho(\chi_2)p_2e_{\alpha_2}\frac{\mathbf{k}^2}{\kappa^2}+O(|\mathbf{k}|^3) \tag{155}$$

while the small-$\mathbf{k}$ expansion of $G_{\rho F^{mc}*\rho F^{mc}}$ has a structure analogous to that of $G_{\{\rho F^{cc}*\rho F^{mc}\}}$ given in Eq. (154),

$$G_{\{\rho F^{mc}*\rho F^{mc}\}}(\mathbf{k},\chi_2)=\rho(\chi_2)p_2e_{\alpha_2}A^{(2)}_{\rm anal}(\mathbf{k})$$
$$\times\int_0^{p_2}\frac{d\tau}{p_2}[e^{i\mathbf{k}\cdot\mathbf{X}_2(\tau)}-1]+O(|\mathbf{k}|^4). \tag{156}$$

As a consequence, the Fourier transforms of $G_\Pi*\Sigma^{**}_D$ and $G_{\Pi*\rho F^{mc}*\rho F^{mc}}$ start at least at order $|\mathbf{k}|^2$ (in fact, at order

$|\mathbf{k}|^3$ in the case of $G_{\Pi * \rho F^{mc} * \rho F^{mc}}$) while $G_{\Sigma_D} * F^{mc}$ starts at order $|\mathbf{k}|^2$. Therefore the first internal screening rule (1) rewritten in Eq. (140) is satisfied, while the second one (2) is also obeyed since $S_{\alpha\gamma}(\mathbf{r})$ is invariant under rotations.

On the other hand, $\int d\chi_a p_a e_{\alpha_a} \rho(\chi_a) f(\mathbf{k},\chi;\chi_a)$, with $f = \Sigma_D$ or $f = \rho F^{cm} * G$, starts at least at order $|\mathbf{k}|$ [in fact at order $|\mathbf{k}|^2$ when $f = \Sigma_D$, according to Eq. (115)]. Eventually, the above decomposition of h exhibits the fact that the small-$\mathbf{k}$ expansion of $\int d\chi_a p_a e_{\alpha_a} G_{\{h-F^{cc}\}}(\mathbf{k},\chi_a)$ starts at order $|\mathbf{k}|^3$ and according to Eq. (143) the external screening rule is satisfied.

3. Large-distance decay

In this section we show that the induced charge density decays with the same power law as the particle-charge correlation. [After only a quick glance at the linear response expression (143), one might have rather thought that the induced charge density should decay as $1/r^{p-2}$ if the particle-charge correlation falls off as $1/r^p$]. The property to be proved means that if the first nonanalytic term in the Fourier transform of the particle-charge correlation is of order $|\mathbf{k}|^{p-3}$, then the first nonanalytic term in $\int d\chi_a p_a e_{\alpha_a} \rho(\chi_a) G_{\{h-F^{cc}\}}(\mathbf{k},\chi_a)$ is equal to a nonanalytic term of greater order, namely, of order $|\mathbf{k}|^{p-1}$. The latter property is nontrivial.

In fact, the dressing devised to prove the external screening sum rule is to be pushed further in order to get the announced result. At the same time we get the diagrammatic structures of the leading tails of the induced charge density. These structures turn out to involve only h^{nn}—with a proper dressing—as the leading tails of the internal correlations.

In the presence of $\mathbf{B}_0$, according to the screening properties (115), (155), and (156), and again since any diagram decays at least as $1/r^5$ after integration over the shapes $\mathbf{X}$'s of the root points [see Eq. (129)], $h_{(B*)}$ proves to be responsible for an algebraic decay in the induced charge density that falls off at least as $1/r^7$ at large distances. By using the same properties [except for Eq. (115)] together with Eqs. (127) and (128) and the right- (left-) dressing relation for h (h^{-n}), we obtain that $h_{(C*)}$ gives a $1/r^5$ tail to the induced charge density. The latter tail comes in fact only from

$$F^{cm} \rho * h^{nn} * [\Sigma_D^{**} + \rho F^{mc} * \rho F^{mc}]. \tag{157}$$

In the absence of $\mathbf{B}_0$, in Sec. IV E of Ref. [5], the right-dressing relations together with the ''screening'' properties (115), (155), and the behaviors (130) and (131) of the decays of diagrams are unchanged when a factor $\int_0^p d\tau \exp[i\mathbf{k}\cdot\mathbf{X}_b(\tau)]$ is introduced. By using them repeatedly, we get that the induced charge density decays as $1/r^8$. Moreover, the latter tail arises only from

$$\Sigma_D^* * h^{nn} * \Sigma_D^{**} \tag{158}$$

as already implicitly shown in Sec. IV E of Ref. [5].

As a conclusion, in the presence of the magnetic field, there is no cascade of power laws for the leading behaviors of the correlations when charges are summed over. In the absence of magnetic field, this cascade is generated by the combination of the remarkable screening property of the Debye polarization cloud and the invariance under rotations. Indeed, the latter allows the harmonicity of the Coulomb potential to play a role: it changes leading tails that would *a priori* decay algebraically as $1/r^5$, $1/r^7$, and $1/r^9$ into short-ranged fall off's, and it induces the special structure of the leading and subleading algebraic tails (130) and (131).

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APPENDIX A

In this appendix we study the first values taken by $P(q,q')$ for any elementary algebraic tail $S^{(\gamma)[q,q']}$ which are used in Sec. VII B. The discussion is analogous to that of Appendix B in Ref. [5]. The differences will be pointed out as comments. The discussion is carried out in two steps. In the first step, we consider the case where $S^{(\gamma)[q,q']}$ comes from a single algebraic bond which is possibly convoluted with fast decaying functions. Then the property (B) defined in Eqs. (88) and (89) is obviously satisfied. In the second step, convolutions involving more than one algebraic bond are considered. We also present results when $S^{(\gamma)[q,q']}$ contains no W bond, because this case corresponds to the algebraic tails of a $\tilde{\Pi}_{W_c}$ diagram with $L=1$, which is considered in Sec. VII B. The details are the following.

If $S^{(\gamma)[q,q']}$ comes from a single bond W then $\gamma = 1+q+q'$, i.e., $P(q,q')=1$. If $S^{(\gamma)[q,q']}$ comes from a single bond F_{R6}, then, $S^{(\gamma)[q,q']} = \Pi_{p=1}^{P_w} w^{[m_p,n_p]}$, with $m_p \geqslant 1$ and $n_p \geqslant 1$ and $P_w \geqslant 2$. So $\gamma = P+q+q'$, with $P = P_w$, $q = \Sigma_{p=1}^{P} m_p$, and $q' = \Sigma_{p=1}^{P} n_p$ and all $P(q,q')$'s such that

$$2 \leqslant P(q,q') \leqslant \inf(q,q') \tag{A1}$$

are realized.

Now, we consider the case in which $S^{(\gamma)[q,q']}$ comes from a single algebraic bond in convolution with two fast-decaying functions $F^{[q,Q_1]}$ and $F^{[Q_1',q']}$ where the various superscripts $[Q,Q']$ have the same meaning as in the definition of property (B). In the following, internal degrees of freedom that are different from the shapes $\mathbf{X}$ are omitted. The expression of $S^{(\gamma)[q,q']}$ in Fourier space reads

$$\begin{aligned} S_{(1)}^{(\gamma-3)[q,q']}(\mathbf{k},\mathbf{Z},\mathbf{Z}') &= \int D(\mathbf{X}_1) \int D(\mathbf{X}_1') F_1^{(n)[q,Q_1]}(\mathbf{k},\mathbf{Z},\mathbf{X}_1) \\ &\quad \times S^{(\gamma_1-3)[q_1,q_1']}(\mathbf{k},\mathbf{X}_1,\mathbf{X}_1') F_2^{(n')[Q_1',q']}(\mathbf{k},\mathbf{X}_1',\mathbf{Z}'). \end{aligned} \tag{A2}$$

$F^{[q,Q_1]}$ and $F^{[Q_1',q']}$ are analytic terms in Fourier space, and in Eq. (A2) n (n') denotes the order of the first term in the small-$\mathbf{k}$ expansion of $F^{[q,Q_1]}$ ($F^{[Q_1',q']}$) that gives a nonvanishing contribution after integration over the shapes $\mathbf{X}_1$ ($\mathbf{X}_1'$) of the intermediate loop. When n

$=0$, $F^{(0)[q,Q_1]}(\mathbf{k},\mathbf{Z},\mathbf{X}_1)=F^{[q,Q_1]}(\mathbf{k}=\mathbf{0},\mathbf{Z},\mathbf{X}_1)$ and when $n=1$, $F^{(1)[q,Q_1]}(\mathbf{k},\mathbf{Z},\mathbf{X}_1)=\int d\mathbf{x}\, i(\mathbf{k}\cdot\mathbf{x})F^{[q,Q_1]}(\mathbf{x},\mathbf{Z},\mathbf{X}_1)$. Thus the order in $|\mathbf{k}|$ of the nonanalytic term (A2) is $\gamma-3$ with

$$\gamma=\gamma_1+n+n', \tag{A3}$$

where n and n' take the value 0 or 1 that is determined by invariance under inversion as follows. We use the invariance of $D(\mathbf{X}_1)$ under $\mathbf{X}_1\to-\mathbf{X}_1$ and the invariance of $F^{[q,Q_1]}(\mathbf{k},\mathbf{Z},\mathbf{X}_1)$ under global inversion of its arguments, as well as the definitions of the superscripts q_1 and Q_1. If $n=0$, $S^{(\gamma-3)[q,q']}(\mathbf{k},-\mathbf{Z},\mathbf{Z}')=(-1)^{q_1}S^{(\gamma-3)[q,q']}(\mathbf{k},\mathbf{Z},\mathbf{Z}')$, while, if $n=1$,

$$S^{(\gamma-3)[q,q']}(\mathbf{k},-\mathbf{Z},\mathbf{Z}')=(-1)^{q_1+1}S^{(\gamma-3)[q,q']}(\mathbf{k},\mathbf{Z},\mathbf{Z}').$$

On the other hand, the definition of the superscript q is $S^{(\gamma-3)[q,q']}(\mathbf{k},-\mathbf{Z},\mathbf{Z}')=(-1)^{q}S^{(\gamma-3)[q,q']}(\mathbf{k},\mathbf{Z},\mathbf{Z}')$. Hence, q and q_1+n have the same parity. Since $\gamma_1=P_1(q_1,q_1')+q_1+q_1'$ with $\inf(q_1,q_1')\geqslant P_1$, we can write $\gamma=P+q+q'$ with $P=P_1(q_1,q_1')$ and $q=q_1+n\geqslant P$ and $q'=q_1'+n'\geqslant P$, so that properties (*A*) and (*B*) are both satisfied. We notice that if $q=1$, then $(n=0,\ q_1=1)$ so that $P=1$. This property will be preserved in the second step of the discussion.

In a second step, we consider the case where $J\geqslant 2$ algebraic bonds are involved in the convolution. After integration over the intermediate points of every product made of a nonanalytic term times an analytic one, such a convolution reads in Fourier space, as in Eq. (B6) of Ref. [5],

$$\begin{aligned}S_{(J)}^{(\gamma)[q,q']}(\mathbf{k},\mathbf{Z},\mathbf{Z}')&=\int\left[\prod_{j=1}^{J-1}d\chi_j\rho(\chi_j)\right]S_{(1)}^{(\gamma_1-3)[q,q_1']}(\mathbf{k},\zeta,\chi_1)\\&\quad\times S_{(1)}^{(\gamma_2-3)[q_2,q_2']}(\mathbf{k},\chi_1,\chi_2)\cdots\\&\quad\times S_{(1)}^{(\gamma_J-3)[q_J,q']}(\mathbf{k},\chi_{J-1},\zeta').\end{aligned} \tag{A4}$$

After integration over the shapes of the internal points χ_j, inversion invariance implies that only even values of $q_j'+q_{j+1}$ do contribute. Henceforth $\gamma=P(q,q')+q+q'$ with

$$P(q,q')=3-3J+\sum_{j=1}^{J}P_j+\sum_{j=1}^{J-1}[P_j+P_{j+1}+\theta(P_j+P_{j+1})]+2N. \tag{A5}$$

One must consider two cases, because the results are not the same in the presence or in the absence of magnetic field.

Case (1). If $P_j=1$ for all $j=1,\ldots,J$ (namely, all nonanalytic terms arise from *W* bonds), the corresponding $S^{(\gamma)[q,q']}(\mathbf{k},\mathbf{Z},\mathbf{Z}')$ comes from a convolution $F_1*w^{[q_1,q_1']}*F_2*w^{[q_2,q_2']}*F_3*\cdots*F_J*w^{[q_J,q_J']}*F_{J+1}$, and the corresponding nonanalytic term reads

$$\begin{aligned}S_{(J)}^{(\gamma)[q,q']}(\mathbf{k},\mathbf{Z},\mathbf{Z}')&\propto\int\prod_{j=1}^{J}D(\mathbf{X}_j)\int\prod_{j=1}^{J}D(\mathbf{X}_j')F_1(\mathbf{k},\mathbf{Z},\mathbf{X}_1)\\&\quad\times\frac{(\mathbf{k}\cdot\mathbf{X}_1)^{q_1}(\mathbf{k}\cdot\mathbf{X}_1')^{q_1'}}{\mathbf{k}^2}F_2(\mathbf{k},\mathbf{X}_1',\mathbf{X}_2)\\&\quad\times\frac{(\mathbf{k}\cdot\mathbf{X}_2)^{q_2}(\mathbf{k}\cdot\mathbf{X}_2')^{q_2'}}{\mathbf{k}^2}\cdots F_J(\mathbf{k},\mathbf{X}_{J-1}',\mathbf{X}_J)\\&\quad\times\frac{(\mathbf{k}\cdot\mathbf{X}_J)^{q_J}(\mathbf{k}\cdot\mathbf{X}_J')^{q_J'}}{\mathbf{k}^2}F_{J+1}(\mathbf{k},\mathbf{X}_J',\mathbf{Z}').\end{aligned} \tag{A6}$$

In the absence of magnetic field, the rotational invariance ensures that, since F_j is an analytic function of $\mathbf{k}$, for $j=1,\ldots,J-1$,

$$\begin{aligned}&\int D(\mathbf{X}_j')\int D(\mathbf{X}_{j+1})(\mathbf{k}\cdot\mathbf{X}_j')^{q_j'}F_{j+1}(\mathbf{k},\mathbf{X}_j',\mathbf{X}_{j+1})\\&\quad\times(\mathbf{k}\cdot\mathbf{X}_{j+1})^{q_{j+1}}\\&=|\mathbf{k}|^{q_j'+q_{j+1}+\theta(q_j'+q_{j+1})}\left[A_{j+1}^{(0)}+\sum_{n=1}^{\infty}A_{j+1}^{(2n)}|\mathbf{k}|^{2n}\right].\end{aligned} \tag{A7}$$

Thus Eq. (A6) may still contain a nonanalytic term, namely, a $1/|\mathbf{k}|^2$ term, after integration over the loop shapes but only in the case $q_j'=q_{j+1}=1$ for $j=1,\ldots,J-1$. In this case $P_j=1$ for all j's and in formula (A5) only the term with $N=0$, i.e., $P(q,q')=1$, corresponds to an algebraic tail while the other values of N correspond to short-ranged decays. Finally, only the value $P(q,q')=1$ is realized when $\mathbf{B}_0=\mathbf{0}$.

In the presence of the magnetic field, the invariance under rotations is broken in one space direction, and in Eq. (A7), in place of $|\mathbf{k}|^{q_j'+q_{j+1}+\theta(q_j'+q_{j+1})}$, there appears a sum of terms $|\mathbf{k}|^{2(N_j-n_j)}[\mathbf{k}]_z^{2n_j}$ with $N_j=q_j'+q_{j+1}+\theta(q_j'+q_{j+1})$ and $n_j=0,\ldots,N_j$. Thus nonanalytic terms of type (85) remain after integration over the loop shapes and in Eq. (A5) all values of N do correspond to some algebraic tails, namely, all values $P(q,q')=1+2N$ with $N\geqslant 0$ are indeed realized.

Case (2). There exists at least one $P_{j_0}\neq 1$. Then, the nonanalyticity is never canceled by the integration over loop shapes, because $S^{(\gamma_{j_0}-3)[q_{j_0},q_{j_0}']}(\mathbf{k},\mathbf{X}_{j_0},\mathbf{X}_{j_0}')$ arises from an F_{R6} bond. The values taken by $P(q,q')$ are determined by inspection, according to the discussion of Appendix B of Ref. [5] just after Eq. (B9). If $q=q'=1$, then $P_1=P_J=1$ and the values given by Eq. (A5) are $P(q,q')=6,7,\ldots.$ ($P(q,q')=6$ [$P(q,q')=7$] is realized when $P_j=1$ for all j's except one j_0 that is different from 1 and J and $P_{j_0}=2$ [$P_{j_0}=3$].) Contrarily, if $q>1$ (or $q'>1$) then P_1 (or P_J) may take the value 2 and the values given by Eq. (A5) are $P(q,q')=4+2N$, so that $P(q,q')=4$ is also realized. [$P(q,q')=4$ is realized when $P_j=1$ for all j's except $j=1$ if either $q>1$ or $j=J$ if $q'>1$.]

As a conclusion, when $\mathbf{B}_0=\mathbf{0}$

$$P(q,q')=\begin{cases}1,6,7,\ldots & \text{if} \quad q=q'=1\\ 1,4,6,7,\ldots & \text{if} \quad (q=1,\ q'\geqslant 2) \ \text{ or } \ (q\geqslant 2,\ q'=1)\\ 1,2,4,5,\ldots & \text{if} \quad \inf(q,q')=2\\ 1,2,\ldots & \text{if} \quad \inf(q,q')\geqslant 3.\end{cases} \tag{A8}$$

When $\mathbf{B}_0\neq\mathbf{0}$, Eq. (B10) of Ref. [5] is still valid (up to a misprint) and reads

$$P(q,q')=\begin{cases}1,3,5,6,\ldots & \text{if} \quad q=q'=1\\ 1,3,4,\ldots & \text{if} \quad (q=1,\ q'\geqslant 2) \ \text{ or } \ (q\geqslant 2,\ q'=1)\\ 1,2,\ldots & \text{if } \inf(q,q')\geqslant 2.\end{cases} \tag{A9}$$

Finally, in view of the discussion in Sec. VII B, we consider a $S^{(\gamma)[q,q']}$ without any W bond. This is the case of a tail T with $L=1$ arising from a $\tilde{\Pi}_{W_c}$ diagram. When only one algebraic bond is involved then it is an F_{R6}, and $2\leqslant P(q,q')\leqslant\inf(q,q')$. In the case of a convolution of $J\geqslant 2$ algebraic bonds without any W bond in the convolution, $P_j\geqslant 2$ for all $j=1,\ldots,J$ and $\inf(q,q')\geqslant 2$. If $P_j=2$ for all j's, then, according to Eq. (A5), $P(q,q')=3J-1+2N$ takes the values $P(q,q')=5+2N$ for $J=2$, $P(q,q')=8+2N$ for $J=3,\ldots$. If $P_j=2$ for all j's except $P_1=3$, then $P(q,q')=3J+2+2N$ takes the values $5+2N$ for $J=2,\ldots$. Finally, when J varies, $P(q,q')=5,7,8,\ldots$. Moreover, the convolutions of case (1) do not exist and the results are the same in the presence or in the absence of magnetic field. As a conclusion,

$$P(q,q';\tilde{\Pi}_{W_c},L=1)=\begin{cases}2,5,7,8,\ldots & \text{if} \quad \inf(q,q')=2\\ 2,3,5,7,8,\ldots & \text{if} \quad \inf(q,q')=3\\ 2,3,4,5,7,8,\ldots & \text{if} \quad \inf(q,q')=4,5\\ 2,3,\ldots & \text{if} \quad \inf(q,q')\geqslant 6.\end{cases} \tag{A10}$$

In fact, the important results for the discussion of the diagrams (91) are the minimal odd and even values given in Eqs. (94) and (96).

APPENDIX B

In this appendix we give the errata for Appendix B of Ref. [5] which deals with the structure of elementary algebraic tails defined in Sec. III D of the latter reference before integration over the loop shapes of their end points. We recall that the definition of these tails is not the same as that given in Sec. VII A of the present paper. The results apply only to the case $\mathbf{B}_0=\mathbf{0}$.

(1) The expression $A^{[q]}_{\mu_1,\ldots,\mu_q}(\mathbf{Z})=[\mathbf{Z}]_{\mu_1}\cdots[\mathbf{Z}]_{\mu_q}f(|\mathbf{Z}|)$ which was written just after Eq. (3.26) and in the second line of Eq. (B2) is not general enough, because the tensor $A^{[q_1'+n]}_{\{\}}(\mathbf{Z}')$ may contain both components of $[\mathbf{Z}]_\nu$ and tensors $\delta_{\nu_i,\nu_{i+1}}$.

(2) After Eq. (B2), the sentence ''The tensor $A^{[q_1'+n]}_{\{\}}(\mathbf{Z}')$ of rank q'_{1+n} is nonzero only if q'_{1+n} is even'' must be replaced by ''The tensor $A^{[q_1'+n]}_{\{\}}(\mathbf{Z}')$ of rank $q_1'+n$ is of parity $(-1)^{q_1'+n}$. As a consequence, $\int D(\mathbf{Z}')A^{[q_1'+n]}_{\{\}}(\mathbf{Z}')$, is nonzero only if $q_1'+n$ is even.''

(3) If $\mathbf{B}_0=\mathbf{0}$, the following modification has to be made. After Eq. (B9), when the P_j's vary the first even value for $P(1,1)$ is 6, and $P(1,1)$ does not take the values with $P=3$ or $P=5$, because the corresponding terms are in fact analytic when $\mathbf{B}_0=\mathbf{0}$; indeed, the $1/\mathbf{k}^2$ singularity is canceled by the property $\int d\chi_j\rho(\chi_j)A^{[1]}_\mu(\mathbf{X}_j)A^{[1]}_\nu(\mathbf{X}_j)\propto\delta_{\mu,\nu}$. (On the contrary, if $\mathbf{B}_0\neq\mathbf{0}$, then $P=3,5$ are realized.) Subsequently, in the same paragraph, $P(q,q')$ may take the values $P(q,q')=1,4,6,7$, as soon $q'\geqslant 2$ or $q\geqslant 2$, and not the values 3 and 5. Eventually, Eq. (B10) must be replaced by Eq. (A8). However, the important result remains the same: the first even values taken by $P(q,q')$ are those given in Eq. (96). If there is no W bond in the convolution, then $q\geqslant 2$ and $q'\geqslant 2$ and the first allowed value for $P(q,q')$ is 2, and Eq. (B11) is to be replaced by Eq. (A10). However, the important result is about the first odd value taken by $P(q,q')$ which is given in Eq. (94). $P_{W_c}(q,q')=2,3,\ldots,\inf(q,q')$ comes from an F_{R6}. The values $5,7,8,\ldots$ are realized by a convolution.

(4) In the case of $\tilde{\Pi}_{W_c}$ diagrams, the value $P(q,q')=3$ was omitted in Eq. (B11) of Ref. [5], though it appears as soon as $\inf(q,q')\geqslant 3$, whether $\mathbf{B}_0=\mathbf{0}$ or $\mathbf{B}_0\neq\mathbf{0}$.

APPENDIX C

In the present appendix we give errata for Appendix C of Ref. [5] which deals with the structure of algebraic tails for various functions involving diagrams $\tilde{\Pi}_{W_c}$. Appendix C of Ref. [5] proves to be valid when $\mathbf{B}_0\neq\mathbf{0}$.

If $\mathbf{B}_0=\mathbf{0}$, then the correct version of Eq. (B10) of Ref. [5] that is given in Eq. (A8) of the present paper causes a modi-

fication of Eqs. (C1), (C3), and (C4), which become, respectively,

$$\delta_{T(\mathrm{I})} = 0,5,\ldots, \tag{C1}$$

$$\delta_{T(\mathrm{II})} = 0,3,5,6,\ldots, \tag{C2}$$

and

$$\delta_{T(\mathrm{III})} = 0,1,3,4,\ldots. \tag{C3}$$

According to the modified version of Eq. (B11) of Ref. [5], namely, Eq. (A10) of the present paper, the correct version of Eq. (C5) is

$$\delta_{T(\mathrm{IV})} = \begin{cases} 0,3,5,6,\ldots & \text{if } \inf(q,q') = 2 \\ 0,1,3,5,6,\ldots & \text{if } \inf(q,q') \geqslant 3. \end{cases} \tag{C4}$$

However, the important result for the discussion of Eq.(92) is that given in Eq. (99). Then, the discussion of Appendix C is unchanged, apart from the following misprint: Eq. (C37) must be replaced by Eq. (C40) of Ref. [5], namely, by Eq. (98) of the present paper.

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